
A Ph.D. THESIS Project

By

Adel Jawahdou

Thesis Advisor

Pr. Abderrazek KAROUI
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Chapter 1

Introduction

Nonlinear integral equations is considered as an important part of the area of nonlinear analysis. It is well known that this subject is a rich source of inspiration in pure and applied mathematics, physics, optimization and control problems, mechanics and engineering, nonlinear programming, economics, finance, transportation and elasticity problems and many other fields [5, 18, 19, 21, 22, 25, 28, 34, 40]. Many problems from nonlinear analysis are often expressed as differential or integro-differential equations and then transformed into integral equations to facilitate their study. The form and properties of the integral operator associated with the integral equation will influence the choice of the space. In the analysis of solution of the equations, these properties may be exploited, and hence, it is often advantageous to consider the nonlinear integral equations obtained as special cases of more general equations in abstract spaces. Starting with the nineteenth century, nonlinear integral equations had the form

\[ x(t) = f(t) + \int_0^t k(t, s, x(s))ds. \]  

(1.1)

and it is known as a Volterra-type integral equation. The functions \( f \) and \( k \) are knowns, while \( x(\cdot) \) is the unknown solution to be found. Such equations do appear in many
applied problems, and their study has been under researchers attention for more than a century. The following references cover most of the aspects encountered in the classic literature, see \[24, 56, 63\]. Another class of integral equations, known as Fredholm-type equations (or Uryshon-type), is represented by

\[
x(t) = f(t) + \int_0^T k(t, s, x(s)) ds,
\]

(1.2)

with \( T > 0 \) a fixed real number. The linear case of (1.2) has conducted Fredholm to the celebrated theory, which is at the origin of modern spectral theory (in functional analysis). For further details concerning equations of the form (1.2) and related forms, see \[25, 71\].

Based on the types of equations briefly described above, one can construct various mixed types of functional equations which are present in the contemporary mathematical or applied science literature. For instance, The following functional equations

\[
x(t) = f(t, x(t), \int_0^t k(t, s, x(s)) ds),
\]

(1.3)

\[
\dot{x}(t) = f(t, x(t), \int_0^t k(t, s, x(s)) ds),
\]

(1.4)

are called respectively functional integral equation and integro-differential equations. They have been investigated by many authors, see \[4, 5, 7, 45, 48, 54\].

The main purpose of this thesis is to study the existence problems of various types of nonlinear integral and functional integral equations in different functional spaces. The existence of solutions of the above general integral equations form (1.3) and (1.4). Such existence can be reduced to the search of fixed points of integral operators associated with these equations. It should be noted here that there are many methods for studying this problem, for example: the approximation method, the variational method, see the
book of E. Zeidler [72] and the topological method which is the aim of our thesis.

The abstract fixed point theory of single-valued mappings has evolved as a nature extension of the corresponding classical theory on Euclidean spaces considered by Browder. This theory has been a popular area of research and has a lot applications in various fields. The best known result in this theory is the Banach contraction mapping principle: "Every contraction self mapping of a complete metric space has a unique fixed point".

It has became a rigorous tool for studying nonlinear Volterra integral equations and nonlinear functional differential equations in Banach spaces. In the last several decades a number of generalizations of Banach principle have appeared in the literature, see [16]. In particular, by Schauder [67] and Leray [55] the theory of nonlinear completely continuous operators forms a very useful tool in the theory of operator equations in Banach spaces. It is very often used in the theory of functional equations [1, 18, 45, 53], including ordinary differential equations, equations with partial derivatives [50], etc...

This theory received a new impetus after the work of G. Darbo (1955) [26] who defined new class of operations which contains completely continuous ones and all contractions as well. His method consisted of using the function called measure of noncompactness defined by K. Kuratowski (1924) [52] on the family of all bounded sets of metric spaces (resp. Banach spaces) see chap 3. Darbo gave his fixed point theorem in this light.

*If $T : C \to C$ is a k-set-contraction ($k \in [0, 1]$) defined on a closed bounded convex subset $C$ of a Banach space $X$, then $T$ has at least one fixed point in $C$.* The Darbo’s fixed point theorem which ensures the existence of a fixed point for so called condensing operators and generalizes both the classical Schauder fixed point principle and (a special variant of) Banach’s contraction mapping principle. Darbo’s theorem is not only of theoretical interest, but has found a wealth of applications in both linear and nonlinear analysis.
Typically, such applications are characterized by some loss of compactness which arises in many fields: integral equations with strongly singular kernels, differential equations over unbounded domains, functional-differential equations of neutral type, linear differential operators, nonlinear superposition operators between various function spaces, initial value problems in Banach spaces, and much more.

Our work contains many contributions to the existence theory for nonlinear integral and differential equations on bounded and unbounded domains, with solutions belonging to the spaces of continuous functions and integrable functions $C(\mathbb{R}_+)$ and $L^p((a,b))$, $p \geq 1$, $(a,b) \subseteq \mathbb{R}_+$, respectively. Let us briefly describe the contents of our thesis without insisting on the exact meaning of the notations.

Chapter 2 is divided into two parts. First, we consider a compact interval $[a,b]$, a positive real number $p \geq 1$ and give an existence result for $L^p$-solutions of the following nonlinear integral equation of the Hammerstein’s type

$$x(t) = g(t) + \int_a^b K(t,s)f(s,x(s))ds, \quad -\infty < a \leq t \leq b < +\infty. \quad (1.5)$$

By using some conditions on the functions $g(\cdot)$ and $f(\cdot, \cdot)$ and by combining Arzela-Ascoli’s theorem with an $L^p-$density result, we prove the compactness of the operator $T$ in the $L^p([a,b]), p \geq 1$, setting. Moreover, by using Leary’s and Shauder’s fixed point theorems, we prove some different existence results for problem (1.5), corresponding to different cases of the kernel $K(\cdot, \cdot)$ and the function $f(\cdot, \cdot)$. These results are obtained in the framework of a joint work with A. Karoui, entitled $L^p$-Solutions of Nonlinear Fredholm Integral Equations, appeared in Applied Mathematics and Computation, see [46].

In the second part, we extend the criteria of compactness and continuity used in the first part to the space $L^p(\mathbb{R}_+, d\mu)$, where $\mu$ is a weight function. Existence results are
given for the nonlinear integral equation

\[ x(t) = g(t) + \int_{0}^{+\infty} K(t,s)f(s,x(s))\,ds. \]  

(1.6)

It is well known that solving the existence problems of such equations on unbounded domains is more challenging than the case of a bounded domain. The treatment of the above nonlinear integral equation with \( f(,..) \) having a polynomial nonlinearity type is in general a difficult problem to solve. In this thesis, we should solve the above problem in the weighted \( L^p \)–space setting. This result is a part of a joint work with A. Karoui and H. Ben Aouicha, entitled Weighted \( L^p \)– Solutions on Unbounded intervals of Nonlinear Integral Equations of the Hammerstein and Urysohn Types, published in Advances in Pure and Applied Mathematics, see [47].

In chapter 3, we present an approach for the existence of solutions in \( L^p(\mathbb{R}_+, p \geq 1) \), of the following nonlinear quadratic integral equation

\[ x(t) = a(t) + x(t) \int_{0}^{+\infty} k(t,s)f(s,x(s))\,ds, \quad t \geq 0 \]  

(1.7)

The interest in the above equation has grown during the past few years. This is due to the wide range of applications of such equations. For example, some problems from vehicular traffic theory, biology or the theory of radiative transfer are modeled by nonlinear quadratic integral equations of such type (see [18, 39, 57]). It is worthwhile mentioning that equation (1.7) is a generalized form of the so-called quadratic integral equation of Chandrasekhar type [22], which can be very often encountered in several applications.

The main ingredients in the existence proofs is based on the concept of measure of noncompactness and the Darbo’s fixed point theorem. The content of this chapter is part of my contribution in the article with A. Karoui and H. Ben Aouicha published in Numerical Functional Analysis and Optimization and entitled: Existence and numerical
solutions of quadratic integral equations defined on unbounded intervals [49].

In chapter 4, we present an existence of solutions of the quadratic integral equation

\[ x(t) = a(t) + \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}} ds, \tag{1.8} \]

in the Fréchet space \( C(\mathbb{R}_+) \). We should mention that the results of this paragraph can be considered as the extension of the existence results given in [57] to the monotonic solutions of quadratic Urysohn’s integral equations of fractional order case. The main tool used in our investigations is a special measure of noncompactness constructed in \( C(\mathbb{R}_+) \) combined with Schauder-Tychonoff fixed point theorem that enable us to study the solvability of considered equations in the class of monotonic functions defined on \( \mathbb{R}_+ \).

This result is a joint work with A. Karoui, appeared in Canadian Applied Mathematics Quarterly and entitled: Monotonic solutions of nonlinear integral equations of fractional order, see [41].

Finally, in chapter 5, we to apply the tool of measure of noncompactness fixed points based theorem to give an alternative approach for the existence of solution of integro-differential evolution equation in Banach space \( E \) of the form

\[
\begin{cases}
  x'(t) = A(t)x(t) + f\left(t, x(t), \int_0^t u(t,s,x(s))ds\right), & t \geq 0, \\
  x(0) = x_0.
\end{cases}
\tag{1.9}
\]

Where \( A(t) : D_t \subset E \to E \) is an infinitesimal generator of an analytic semigroup of bounded linear operators \( U(t,s) \) and \( f : \mathbb{R}_+ \times E \to E \) is a given function.
Chapter 2

$L^p$—Existence solutions of nonlinear integral equations of the Hammerstein and Urysohn types

In this chapter, we give various existence results of solutions of nonlinear integral equation of the Hammerstein and Urysohn types. These solutions are given in the framework of either the $L^p([a,b])$ or the weighted $L^p(\mathbb{R}_+)$ spaces, with $1 \leq p < +\infty$.

2.1 Background and Existent Results

Let $\Omega$ be a subset of the $n$-dimensional Euclidean space $\mathbb{R}^n$ and let $X$ and $Y$ denote Banach spaces of real-valued measurable functions on $\Omega$. A nonlinear integral equation of Hammerstein type in the Banach space $X$ is an equation of the form

$$x(t) + \int_{\Omega} k(t, s)f(s, x(s))ds = y(t),$$
for a given function $y$ in $X$ and unknown function $x$ in $X$. Here $k(t, s)$ is a real-valued measurable function on $\Omega \times \Omega$ such that the linear operator $A$ defined by

$$Ax(t) = \int_{\Omega} k(t, s)x(s)ds$$

is well defined from the Banach space $Y$ into the Banach space $X$. In the special case where $\omega = (a, b), \infty \leq a < b \leq +\infty$, the above integral equation is a special case of the following Urysohn type integral equation,

$$x(t) = f(t) + \int_{a}^{b} u(t, s, x(s))ds, \quad -\infty \leq a \leq b \leq +\infty \quad (2.1)$$

Usually the proof of the existence of a solution of (2.1) starts with some conditions on the function $u(t, s, x)$ as well as the limits of integration $a, b$ and the function $f(.)$. Based on these conditions, a Banach space is chosen in such a way that the existence problem is converted into a fixed-point problem for an operator over this Banach space. Many authors are discussed the solvability of (2.1) in some Banach space $X$, under some assumptions and by using various fixed point theorems. In particular, in [11, 45] the authors solved (2.1) in the space of $X = C([a, b]), -\infty < a \leq b < +\infty$ and $L^1[0, 1]$ respectively. Their Theorems of existence are given as follows

**Theorem 2.1 (Theorem 5 in [47]).** Assume that the function $f(.) \in C([a,b])$ and $u(t, s, x)$ satisfies the following conditions:

(i) $\sup \left( |u(t, s, x)|, |\frac{\partial u}{\partial t}(t, s, x)| \right) \leq V_1(t)V_2(s)\phi(|x|)$,

(ii) $|\frac{\partial u}{\partial t}(t, s, x)| \leq V_1(t)V_2(s)\psi(|x|)$,

where $V_1(.) \in C([a,b]), V_2 \in L^1([a,b]), \phi(.)$ is positive and bounded over $[0, +\infty[$ and $\psi(.)$ is positive and continuous over $[0, +\infty[$. Under the above conditions, (2.1) has a solution in $C([a,b])$.  

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Theorem 2.2 (Theorem 5.1 in \[11\]). Assume that

\( h_1 \). \( u(t, s, x(s)) = u : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R} \) is measurable on \([0, 1]\) for all \( s \in [0, 1] \) and \( x \in \mathbb{R} \). Moreover, the function \((s, x) \to u(t, s, x)\) is continuous on the set \([0, 1] \times \mathbb{R}\) for each \( t \in [0, 1] \),

\( h_2 \). \( f \in L^1[0, 1], \)

\( h_3 \). \( |u(t, s, x)| \leq k(t, s)(a(s) + b|x|) \) for \((t, s, x) \in [0, 1] \times [0, 1] \times \mathbb{R}, \) where \( b \) is a nonnegative constant, \( a \in L^1[0, 1] \) and \( a(t) > 0 \) for almost all \( t \in [0, 1] \). Apart from this we assume that \( k : [0, 1] \times [0, 1] \to [0, \infty) \) satisfies the Caratheodory conditions and is such that the integral operator \( K \) generated by \( k \) i.e.

\[ (Kx)(t) = \int_0^1 k(t, s)x(s)ds, \quad (t \in [0, 1]) \]

is a continuous mapping of \( L^1[0, 1] \) into itself for which \( b\|K\| < 1 \), where \( \|K\| \) denotes the norm of the operator \( K \).

Then, there exists at least one solution of the equation (2.1) such that \( x \in L^1[0, 1] \).

Note that the existence result given by the first theorem is based on the use of Shaefers fixed point theorem. The existence results of the second theorem is based of the use of an appropriate fixed point theorem associated with a measure of noncompactness.

In this chapter, we shall extend the above existence results to more general functional spaces. Precisely, we shall give a new existence in the spaces \( L^p_\mu([a, b]) \), where \( 1 \leq p < \infty, -\infty \leq a \leq b \leq +\infty \) and \( \mu \) is an weighted given function. Moreover, by using Schauders and Shaefers fixed point theorems, we prove different existence results for problem

\[ x(t) = a(t) + \int_0^{+\infty} k(t, s)h(s, x(s))ds, \quad t \geq 0, \quad (2.2) \]
corresponding to different cases of the kernel $k(\cdot,\cdot)$ and the nonlinear operator $h(\cdot,\cdot)$
growth is polynomial. Moreover, we give an extension of these results to the following
more general case of Urysohn's integral equation

$$x(t) = g(t) + \int_{0}^{+\infty} f(t, s, x(s)) ds.$$ \hspace{1cm} (2.3)

We first consider the nonlinear integral equation of the form

$$x(t) = g(t) + \int_{a}^{b} K(t, s) f(s, x(s)) ds, \hspace{0.5cm} -\infty < a \leq t \leq b < +\infty.$$ \hspace{1cm} (2.4)

It is well known that the fixed point theorems in Banach spaces are considered as powerful
tools for solving the existence problems associated with integral equations. Most of these
fixed point theorems are based on the concept of compact operators acting on some
appropriate Banach spaces. In the special case where $X = C([a, b])$, the Arzela-Ascoli's
theorem allows us to prove the compactness of the operator $T$ defined on $X$ by

$$Tx(t) = g(t) + \int_{a}^{b} K(t, s) f(s, x(s)) ds, \hspace{0.5cm} \forall t \in [a, b].$$ \hspace{1cm} (2.5)

By using some conditions on the functions $g(\cdot)$ and $f(\cdot, \cdot)$ and by combining Arzela-
Ascoli's theorem with an $L^p$–density result, we prove the compactness of the operator
$T$ in the $L^p([a, b]), p \geq 1$, setting. Moreover, by using Shafer's and Schauder's fixed
point theorems, we prove different existence results for problem (2.5), corresponding to
different cases of the kernel $K(\cdot, \cdot)$ and the function $f(\cdot, \cdot)$.

## 2.2 Mathematical Preliminaries

In this section, we give a brief description of Shaefer’s fixed point theorem as well as
the powerful Schauder’s fixed point theorem. Then, we list some properties Nemytskii’s
operator. Also, we give some compactness results in the $L^p$–spaces.
2.2.1 Classical and non classical fixed point theorems in Banach spaces.

For the sake of completeness, we give the following definitions and theorems concerning compact operators and fixed point theorems.

**Definition 2.1** (compact operators, completely continuous operators): Let $X$ be a Banach space and let $T : X \rightarrow X$ be an operator. $T$ is said to be compact, if it maps bounded sets of $X$ into relatively compact sets. Moreover, $T$ is said to be completely continuous, if its is continuous and compact.

In the special case where $X = C([a,b])$, then the following Arzela-Ascoli’s theorem is generally used to prove the compactness of $T$.

**Theorem 2.3** (Arzela-Ascoli’s theorem): A necessary and sufficient condition that a family of continuous functions defined on the compact interval $[a,b]$ be compact in $C([a,b])$ is that this family be uniformly bounded and equicontinuous.

The main tools of our existence results are the following classical fixed point theorems that can be found in [68].

**Theorem 2.4** (Schaefer’s fixed-point theorem): Let $X$ be a Banach space and let $T : X \rightarrow X$ be a completely continuous operator. Then either:

(i) The operator equation $x = \lambda Tx$ has a solution for $\lambda = 1$.

or

(ii) The set $E = \{x \in X; \ x = \lambda Tx, \ \lambda \in ]0,1[\}$ is unbounded.

**Theorem 2.5** (Schauder’s fixed-point theorem): Let $K$ be a closed convex subset of a Banach space $X$. If $T : K \rightarrow K$ is continuous and $K = \overline{T(K)}$ is compact, then $T$ has a fixed point in $K.$
Recently, there is a growing interest in establishing new powerful fixed point theorems that eventually will have important applications in the area of nonlinear integral equation. Some of these new and non classical fixed point theorems are given in [30, 31, 37]. In particular, the following non classical fixed point theorems have been given in the previous references.

Let $\mathbb{R}_+$ be the set of nonnegative real numbers and $F$ the family of mappings $\varphi$ from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that each $\varphi$ is upper semicontinuous, nondecreasing and $\varphi(t) < t$ for all $t > 0$.

**Theorem 2.6** *(Theorem 1 in [37])* Let $C$ be nonempty closed convex subset of a Banach space $X$ and $T$ be a mapping of $C$ into itself satisfying the inequality

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|,$$

for all $x, y$ in $C$, where $a > 0$, $b, c \geq 0$, $a + b + c = 1$. Then, $T$ has a unique fixed point.

**Theorem 2.7** *(Theorem 3 in [31])* Let $A$, $B$, $S$ and $T$ be mappings from a metric space $(X, d)$ into itself satisfying

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X)$$

and

$$\left( \int_0^{d(Ax, Ay)} \psi(t)dt \right)^p \leq \varphi\left( a\left( \int_0^{d(Sx, Ty)} \psi(t)dt \right)^p + (1 - a) \max\left\{ \int_0^{d(Ax, Sx)} \psi(t)dt, \int_0^{d(By, Ty)} \psi(t)dt, \left( \int_0^{d(Ax, Sx)} \psi(t)dt \right)^2, \left( \int_0^{d(Ax, Ty)} \psi(t)dt \right)^2, \left( \int_0^{d(Sx, By)} \psi(t)dt \right)^{\frac{1}{2}}, \left( \int_0^{d(Ax, Ty)} \psi(t)dt \right)^{\frac{1}{2}}, \left( \int_0^{d(Sx, By)} \psi(t)dt \right)^{\frac{1}{2}} \right\} \right).$$

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for all $x, y$ in $X$, where $0 < a \leq 1$, $p \geq 1$ and $\psi$ is a Lebesgue integrable mapping which is summable nonnegative and satisfies to

$$\int_0^\epsilon \psi(t)dt > 0 \text{ for each } \epsilon > 0.$$ 

Suppose that one of $S(X)$ or $T(X)$ is complete and the pairs $(A, S)$ and $(B, T)$ are weakly compatible. Then, $A, B, S$ and $T$ have a unique common fixed point in $X$.

Note that $S$ and $T$ are said to be weakly compatible if they commute at their coincidence points; i.e., if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

### 2.2.2 Nemytskii’s operator

The Nemytskii’s operator plays an important role in the existence problems of nonlinear integral equation of the Hammerstein’s type. Following, we give the definition and some important properties of this operator.

**Definition 2.2** Let $\Omega \subset \mathbb{R}^m$ be an open set and $f : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a given function. The Nemytskii’s operator associated with $f$ assigns to each function $x : \Omega \rightarrow \mathbb{R}^m$, the function $N_f : \Omega \rightarrow \mathbb{R}^n$, defined by

$$N_f x(t) = f(t, x(t)) \quad (x \in \Omega).$$

The Caratheodory’s condition is generally used to prove many existence results associated with some nonlinear integral equations, see [15]. The $L^q$– version of the Caratheodory’s condition is given by the following definition.

**Definition 2.3** ($L^q$–Caratheodory’s condition) Let $q$ be a real number satisfying $q \geq 1$. A mapping $f : [a, b] \rightarrow \mathbb{R}$ is said to satisfy the $L^q$–Caratheodory’s condition if it satisfies
the following three conditions.

(i) The map \( s \to f(s,x) \) is measurable for each \( x \in \mathbb{R} \).

(ii) The map \( x \to f(s,x) \) is continuous almost everywhere \( s \in [a,b] \).

(iii) For each \( r > 0 \), there exists a function \( h_r \in L^q([a,b], \mathbb{R}) \) such that

\[
|f(s,x)| \leq h_r(s), \quad \forall |x| \leq r, \quad \text{a.e. } s \in [a,b].
\]

**Theorem 2.8** Let \( G \) be a measurable subset of \( \mathbb{R} \) and let \( f : G \times \mathbb{R} \to \mathbb{R} \) be a function satisfying the Carathéodory’s conditions i.e.

(i) \( f(\cdot, x) : G \to \mathbb{R} \) is measurable for each \( x \in \mathbb{R} \),

(ii) \( f(t, \cdot) : \mathbb{R} \to \mathbb{R} \) is continuous for a.e. \( t \in G \).

Let \( 1 \leq p, r < \infty \), \( a \in L^r(G) \), and assume that

\[
|f(t,x)| \leq c|x|^{p/r} + a(t), \quad \text{a.e. } t \in G, \quad x \in \mathbb{R}.
\]  

(2.6)

Then the operator defined by

\[
N_f x(t) = f(t,x(t)), \quad \text{a.e. } t \in G, \quad x \in L^p(G)
\]

is bounded and continuous from \( L^p(G) \) to \( L^p(G) \).

The reader is referred to [32], for the proof of the above theorem.

### 2.2.3 Continuity and Compactness Results

We first state the following Vitalli’s lemma which is useful in proving the continuity of integral operators.
**Lemma 2.1** (Vitali’s lemma) Let \((x_n)\) be a sequence of functions \(x_n \in L^p(\mathbb{R}+)\), \((1 \leq p \leq \infty)\) such that \(x_n(t) \to x(t)\) as \(n \to \infty\) for a.e \(t \in \mathbb{R}+\). Then \(x \in L^p(\mathbb{R}+)\) and \(x_n \to x\) in \(L^p(\mathbb{R}+)\) if and only if

(i) for each \(\epsilon > 0\), there exists \(\delta > 0\) such that

\[ \int_A |x_n(t)|^p dt < \epsilon \]

for all \(n\) and every measurable subset \(A \subset \mathbb{R}+\) with \(\text{meas}(A) < \delta\).

(ii) for each \(\epsilon > 0\), there exists a subset \(D \subset \mathbb{R}+\) such that \(\text{meas}(D) < \infty\) and

\[ \int_{\mathbb{R}+ \setminus D} |x_n(t)|^p dt < \epsilon \]

for all \(n\).

**Remark 2.1** Notice condition (ii) in Vitali’s lemma is redundant if we change \(\mathbb{R}+\) by a compact interval \([a, b]\).

**Lemma 2.2** Under the above notations, let \(p \geq 1\) be a real number and let \(q \in [1, +\infty]\), be the conjugate of \(p\). Assume that the superposition operator \(F\) satisfies the following condition,

\[ \forall x \in L^p([a, b]), \quad F(x) \in L^q([a, b]). \]

Then, \(F\) is a continuous mapping from \(L^p([a, b])\) into \(L^q([a, b])\) which maps bounded sets of \(L^p([a, b])\) into bounded sets of \(L^q([a, b])\). Moreover, there exists a constant \(C > 0\) and \(h \in L^q([a, b])\) such that

\[ |f(s, x)| \leq C|x|^{p-1} + h(s), \quad \forall s \in [a, b], \forall x \in \mathbb{R}. \quad (2.7) \]
Next, we give some compactness results for subsets of the space $L^p(X, \Omega)$, where $X$ is a Banach space and $\Omega$ is an open set of $\mathbb{R}^n$.

**Theorem 2.9** (*Riesz Compactness Criteria [57]*). Let $M \subset L^p([a, b], \mathbb{R})$, $1 \leq p \leq \infty$. Then $M$ is relatively compact if and only if the following hold:

i- $M$ is bounded in $L^p([a, b]),$

ii- $\int_a^b |x(t + h) - x(t)|^p dt \to 0$ as $h \to 0$.

If $\Omega$ is not necessarily a compact subset of $\mathbb{R}$, then the following theorem provides us with a compactness criteria of a subset $M$ of $L^p(\Omega)$.

**Theorem 2.10** (*[32]*). A subset $M \subset L^p(\Omega)$ is relatively compact if and only if the following conditions hold:

i- $\sup_{x \in M} \|x\| = \sup_{x \in M} (\int_\Omega |x(s)|^p ds)^{1/p} < \infty,$

ii- $\lim_{h \to 0} \int_\Omega |x(s + h) - x(s)|^p ds$ uniformly in $x \in M$

iii- $\lim_{\delta \to \infty} \int_{\Omega \setminus |s| > \delta} |x(s)|^p ds = 0$ uniformly in $x \in M$.

The following compactness result of an integral operator acting on $L^p([a, b], \mathbb{R})$ has been borrowed from [48]. Note that in the very special case where $f(s, x) = x$, then some $L^p$ compactness results can be found in [8].

**Theorem 2.11** Let $p \geq 1$ be a positive real number and let $q$ the conjugate of $p$. Let $T$ be the operator given by (2.5). Assume that $g \in L^p([a, b])$ and assume that there exists a positive constant $C$ and a function $h \in L^q([a, b])$, such that

$$|f(s, x)| \leq C|x|^{p-1} + h(s), \quad \forall \ x \in \mathbb{R}, \ \forall \ s \in [a, b].$$

(2.8)
Assume that the kernel $K(\cdot, \cdot) \in L^p([a, b]^2)$. Then $\forall x(\cdot) \in L^p([a, b])$, $Tx(\cdot) \in L^p([a, b])$ and $T$ is a compact operator.

**Proof:** We first write the operator $T$ as follows, $T = T_1 + T_K$, where $T_1$ and $T_K$ are defined on $L^2([a, b])$ by

$$T_1(x)(t) = g(t), \quad T_K(x)(t) = \int_a^b K(t, s) f(s, x(s)) \, ds, \quad t \in [a, b].$$

It is clear that $T_1$ is a compact operator on $L^p([a, b])$. Hence, to prove that $T$ is compact, it suffices to prove that $T_K : L^p([a, b]) \to L^p([a, b])$ is compact. Note that from (2.8), one concludes that if $x \in L^p([a, b])$, then the function $s \to f(s, x(s))$ belongs to $L^q([a, b])$.

Let $x \in L^p([a, b])$, then by using the Hölder’s inequality, one gets

$$\|T_K(x)\|_p^p \leq \int_a^b \left( \int_a^b |K(t, s)|^p \, ds \left[ \int_a^b |f(s, x(s))|^q \, ds \right]^\frac{p}{q} \right) dt \leq \|K\|_p^p \left( \|C|x(s)|^{p-1} + \|h(s)\|_q \right)^p \leq \|K\|_p^p \left( C(\|x\|_p)^{p-1} + \|h\|_q \right)^p.$$

Consequently, we have

$$\|T_K(x)\|_p \leq \|K\|_p \left( C(\|x\|_p)^{p-1} + \|h\|_q \right) < +\infty. \quad (2.9)$$

Hence, $\forall x \in L^p([a, b])$, we have $T_K(x) \in L^p([a, b])$.

The proof of the compactness of $T_K$ is done by the following two steps.

**First step:** In this step, we assume that $K(t, s) \in C([a, b]^2)$ and we prove that $T_K : L^p([a, b]) \to C([a, b])$ is compact. We first show that $T_K(L^p([a, b])) \subset C([a, b])$. Let
Note that in this case, there exists a sequence of kernels \((K_n(t, s))_n \in C([a, b])\), such that \(\|K_n - K\|_p \to 0\) as \(n \to +\infty\). Let \(S\) be the previously defined bounded set of \(L^p([a, b])\). Since \(K_1(\cdot, \cdot) \in C([a, b]^2)\), then by using the first step, one concludes that \(T_{K_1}\) is compact. Hence, there exists \((x_n^{(1)})_n\) a subsequence of \((x_n)_n\) such that \(\left(T_{K_1}(x_n^{(1)})\right)_n\) is

Second step: We prove that \(T_K\) is compact in the general case where \(K \in L^p([a, b]^2)\). Note that in this case, there exists a sequence of kernels \((K_n(t, s))_n \in C([a, b]^2)\), such that \(\|K_n - K\|_p \to 0\) as \(n \to +\infty\). Let \(S\) be the previously defined bounded set of \(L^p([a, b])\). Since \(K_1(\cdot, \cdot) \in C([a, b]^2)\), then by using the first step, one concludes that \(T_{K_1}\) is compact. Hence, there exists \((x_n^{(1)})_n\) a subsequence of \((x_n)_n\) such that \(\left(T_{K_1}(x_n^{(1)})\right)_n\) is
convergent. Similarly, there exists \((x_n^{(2)})_n\) a subsequence of \((x_n^{(1)})_n\) such that \((T_{K_2}(x_n^{(2)}))_n\) is convergent. More generally, \(\forall m \in \mathbb{N}\), there exists a subsequence \((x_n^{(m)})_n\) of \((x_n^{(m-1)})_n\) such that \((T_{K_m}(x_n^{(m)}))_n\) is convergent. Next, consider the diagonal subsequence \((x_n^{(n)})_n\), we prove that \((T_K(x_n^{(n)}))_n\) is a Cauchy sequence in \(L^p([a, b])\). We first note that \(\forall k, l, n \in \mathbb{N}\), we have
\[
\|T_K(x_k^{(k)}) - T_K(x_l^{(l)})\|_p \leq \|T_K(x_k^{(k)}) - T_{K_n}(x_k^{(k)})\|_p + \|T_{K_n}(x_k^{(k)}) - T_{K_n}(x_l^{(l)})\|_p + \|T_{K_n}(x_l^{(l)}) - T_K(x_l^{(l)})\|_p.
\]
(2.10)

Since \((T_{K_n}(x_l^{(l)}))_l\) is convergent, then \(\forall \epsilon > 0\), there exists \(N_\epsilon \in \mathbb{N}\) such that
\[
\|T_{K_n}(x_k^{(k)}) - T_{K_n}(x_l^{(l)})\|_p < \frac{\epsilon}{3}, \quad \forall l, k \geq N_\epsilon.
\]
(2.11)

On the other hand, we have
\[
\|T_K(x_k^{(k)}) - T_{K_n}(x_k^{(k)})\|_p \leq \int_a^b \left[ \int_a^b |K_n(t, s) - K(t, s)| f(s, x_k^{(k)}(s)) \right]^p ds \] \(dt \leq \|K - K_n\|_p (C(\|x_k^{(k)}\|_p)^{p-1} + \|h\|_q)^p.
\]
Since \(\forall k \in \mathbb{N}\), \(x_k^{(k)} \in S\), then the previous inequality implies
\[
\|T_K(x_k^{(k)}) - T_{K_n}(x_k^{(k)})\|_p \leq \|K - K_n\|_p (CM_S^{p-1} + \|h\|_q).
\]
(2.12)

since, \(\|K_n - K\|_p \to 0\) as \(n \to +\infty\), then by using the previous inequality, one concludes that there exists \(M_\epsilon \in \mathbb{N}\), such that
\[
\|T_K(x_k^{(k)}) - T_{K_n}(x_k^{(k)})\|_p < \frac{\epsilon}{3}, \quad \forall n \geq M_\epsilon, \forall k \in \mathbb{N}.
\]
(2.13)

By combining (2.10), (2.11) and (2.12), one concludes that \((T_K(x_n^{(n)}))_n\) is a Cauchy sequence in the Banach space \(L^p([a, b])\). Hence, any bounded sequence of \(T_K(S)\) has a convergent subsequence. Consequently, \(T_K(S)\) is compact, whenever \(S\) is a bounded subset of \(L^p([a, b])\). This shows that \(T_K\) is a compact operator over \(L^p([a, b])\).
**Remark 2.2** The technique of the construction of the diagonal Cauchy sequence of Step 2 of the previous proof, has been already used in [51] in the linear integral operators setting.

### 2.3 $L^p$—existence solutions of nonlinear integral equations of the Hammerstein types on bounded intervals.

The first existence result for problem (2.4) is given by the following theorem. The proof of this theorem has been given in [46]. For the sake of convenience, we give the proof here.

**Theorem 2.12** Consider the nonlinear integral equation (2.4). Let $1 \leq p \leq 2$ be a real number and let $q \in [1, \infty]$, be the conjugate of $p$. Assume that $K(\cdot, \cdot) \in L^p([a, b]^2)$ and $g(\cdot) \in L^p([a, b])$ and that the function $f(\cdot, \cdot)$ satisfies the condition of lemma 2.2. Then the following results hold.

1. **(R$_1$)** If $1 \leq p < 2$, then (2.4) has a solution $x \in L^p([a, b]).$

2. **(R$_2$)** If $p = 2$ and the kernel $K$ satisfies one of the following two conditions:
   
   (c$_1$) $C\|K\|_2 < 1$, where the constant $C$ is as given by Lemma 2.2.
   
   (c$_2$) $K(t, s) = 0, \forall s \geq t$ and $|K(t, s)| \leq |K_1(t)||K_2(s)|$, where $K_1(\cdot)$ is bounded and measurable over $[a, b]$ and $K_2(\cdot) \in L^p([a, b])$.

Then, (1.5) has a solution $x \in L^p([a, b]).$

**Proof:** We first note that since the map $f(\cdot, \cdot)$ satisfies the conditions of lemma 1, then it satisfies the inequality (2.7) for some constant $C > 0$ and $h(\cdot) \in L^q([a, b])$. Hence, by
theorem 2.11, one concludes that the operator $T$ defined by (2.5) is a compact operator from $L^p([a, b])$ into $L^p([a, b])$. Moreover, the $L^p$-continuity of $T$ is a simple consequence of the continuity of the superposition operator $F$ over $L^p([a, b])$. More precisely, Let $x \in L^p([a, b])$ and let $(x_n)_n$ be a sequence in $L^p([a, b])$ converging to $x$. Then, by using the properties of $F$ as well as Hölder’s inequality, one obtains

$$\|Tx_n(\cdot) - Tx(\cdot)\|^p_p \leq \int_a^b \left[ \int_a^b |K(t, s)||F(x_n)(s) - F(x)(s)|\,ds \right]^p dt \leq \int_a^b \left[ \int_a^b |K(t, s)|^p\,ds \right]^p \left[ \int_a^b |F(x_n)(s) - F(x)(s)|^q\,ds \right]^{p/q} dt \leq \|F(x_n) - F(x)\|_q \|K\|_p^p. \tag{2.14}$$

Since $F : L^p([a, b]) \to L^q([a, b])$ is continuous, then the previous inequality implies the continuity of $T$. Since, $T$ is already compact, then $T$ is completely continuous. Moreover, if $x \in L^p([a, b])$, then by using again (2.8) together with Hölder’s inequality, one can easily check that

$$\|Tx\|_p \leq \|g\|_p + \|K\|_p \left(C(\|x\|_p)^{p-1} + \|h\|_q\right). \tag{2.14}$$

To prove the existence result ($R_1$), we use Shaefer’s fixed point theorem and prove that for $1 \leq p < 2$, the set $\mathcal{E} = \{x \in L^p([a, b]); \ x = \lambda Tx, \ \lambda \in ]0, 1[\}$ is bounded. By using (2.14), it is easy to see that $\forall x \in \mathcal{E}$, we have

$$\|x\|_p \leq \|Tx\|_p \leq C\|K\|_p (\|x\|_p)^{p-1} + \|g\|_p + \|K\|_p \|h\|_q,$$

or equivalently

$$(\|x\|_p)^{p-1} \left((\|x\|_p)^{2-p} - C\|K\|_p^p\right) \leq \|g\|_p + \|K\|_p \|h\|_q.$$ 

Since $p - 1, 2 - p \geq 0$, then the previous inequality implies that there exists a constant $M > 0$ such that $\|x\|_p \leq M$. Hence, $\mathcal{E}$ is uniformly bounded by $M$. Consequently by using
Shaefer’s theorem, one concludes that (2.4) has always an \( L^p([a,b]) \)-solution, whenever \( 1 \leq p < 2 \). Next, we prove \((R_2)\). We first consider the special case \((c_1)\). Since \( x \in E \), then from (2.14), we have
\[
\|x\|_2 \leq \|Tx\|_2 \leq \|g\|_2 + \|K\|_2 (\|x\|_2 + \|h\|_2).
\]
Since \( C\|K\|_2 < 1 \), then the previous inequality implies that \( \forall x \in E \), we have
\[
\|x\|_2 \leq \frac{\|g\|_2 + \|K\|_2\|h\|_2}{1 - C\|K\|_2} = M.
\]
Hence, \( E \) is bounded. Consequently, if \( C\|K\|_2 < 1 \), then an \( L^2 \)-solution of (2.4) is ensured by Shaefer’s fixed point theorem. Finally, under condition \((c_2)\), any solution of \( x = \lambda Tx \) for some \( \lambda \in [0,1] \) has to satisfy the following inequalities
\[
|x(t)| \leq \|g\|_{\infty} + \|K_1(t)\| \int_a^t |K_2(s)||h(s)| ds + C|K_1(t)| \int_a^t |K_2(s)||x(s)| ds, \quad t \in [a,b]
\]
\[
\leq \|g\|_{\infty} + \|K_1\|_{\infty}\|K_2\|_2\|h\|_2 + C\|K_1\|_{\infty}\int_a^t |K_2(s)||x(s)| ds \leq A + \int_a^t \phi(s)|x(s)| ds,
\]
where \( A = \|g\|_{\infty} + \|K_1\|_{\infty}\|K_2\|_2\|h\|_2 \) and \( \phi(s) = C\|K_1\|_{\infty}|K_2(s)| \in L^2([a,b]) \subset L^1([a,b]) \). Hence, by using Gronwall’s inequality, one concludes that \( x \) satisfies the following a priori bound
\[
\|x\|_2 \leq \sqrt{b-a} A \exp(\|\phi\|_1).
\]
Consequently, an \( L^2([a,b]) \)-solution of (2.4) is ensured by Shaefer’s fixed point theorem. \( \square \)

We should note that the result of the above theorem is valid only in the case where \( 1 \leq p \leq 2 \). Moreover, if \( p = 2 \), then condition \((c_1)\) is a serious limitation of theorem 5. To overcome these problems, one can use a convenient weighted \( L^p \)-norm and Shauder’s fixed point theorem. This is the subject of the following theorem. The proof of this theorem has been given in [46]. For the sake of convenience, we give the proof here.
Theorem 2.13  Consider a real number \( p \geq 1 \) and let \( q \) be the conjugate of \( p \). Consider the integral equation

\[
x(t) = g(t) + \int_a^b K(t,s) f(s,x(s)) \, ds, \quad t \in [a,b],
\]

where the functions \( g(t) \), \( K(t,s) \) and \( f(s,x) \) are as given by the previous theorem. Also assume that there exists a constant \( C_1 > 0 \) and \( h \in L^p([a,b]) \) such that

\[
|f(s,x)| \leq C_1|\mu| + |h(s)|, \quad \text{a.e. } s \in [a,b], \quad \forall x \in \mathbb{R}.
\]

Moreover, assume that there exists a function \( \mu \), continuous, positive on \([a,b]\), bounded away from zero, and such that the function

\[
\Psi(t) = \left[ \int_a^b |K(t,s)|^q(\mu(s))^{-q/p} \, ds \right]^{1/q},
\]

belongs to \( L^p([a,b],d\mu) \). If \( C_1 \|\Psi\|_{p,\mu} < 1 \), then (2.15) has a solution \( x \in L^p([a,b]) \).

Proof: We first define the weighted \( L^p \) – space \( X = L^p([a,b],d\mu) \), by

\[
L^p([a,b],d\mu) = \{ f \in L^p([a,b]), \| f \|_{p,\mu} < +\infty \}.
\]

Here, \( \| \cdot \|_{p,\mu} \) is the positive real valued function defined on \( L^p([a,b]) \) by

\[
\forall f \in L^p([a,b]), \quad \| f \|_{p,\mu} = \left( \int_a^b |f(t)|^p \mu(t) \, dt \right)^{1/p}.
\]

By using the properties of the function \( \mu \), it is clear that \( \| \cdot \|_{p,\mu} \) is a norm and that \( X = L^p([a,b],d\mu) \) is a Banach space. Moreover, it can be easily checked that the two norms \( \| \cdot \|_p \) and \( \| \cdot \|_{p,\mu} \) are equivalent. Hence, a bounded set of \( X \) is also a bounded set of \( L^p([a,b]) \). Next, let \( R \) be a positive real number to be fixed in the sequel and consider the closed, convex and bounded set \( B_R \) of \( X \), given by

\[
B_R = \{ f \in X, \| f \|_{p,\mu} \leq R \}.
\]
Since the integral operator $T$ given by (2.5) is compact on $L^p([a, b])$ and since $B_R$ is a bounded set of $L^p([a, b])$, then $T(B_R)$ is relatively compact in $L^p([a, b]) \subseteq X$. Next, we prove that if $C_1 \|\Psi\|_{p, \mu} < 1$, then there exists $R_0 > 0$ such that $\forall R \geq R_0$, we have $T(B_R) \subseteq B_R$. This is done as follows. Let $x \in B_R$, then by using Hölder’s inequality, (2.16) and Fubini-Tonelli’s theorem, one gets the following inequalities.

\[
\|T_K x\|_{p, \mu}^p \leq \int_a^b \mu(t) \left[ \int_a^b \frac{|K(t, s)|}{\mu(s)} (\mu(s))^{1/p} (C_1 |x(s)| + |h(s)|) \frac{\mu(s)^{1/p}}{\mu(s)^{1/q} \|x\|_{p, \mu} + \|h\|_{p, \mu}} \right] dt
\]

Since $T x = g + T_K x$, and since $\|x\|_{p, \mu} \leq R$, then the previous inequality implies

\[
\|T x\|_{p, \mu} \leq \|g\|_{p, \mu} + \|\Psi\|_{p, \mu} \|h\|_{p, \mu} + C_1 \|\Psi\|_{p, \mu} \|x\|_{p, \mu}.
\]

Consequently, $T(B_R) \subseteq B_R$, whenever

\[
R \geq \frac{\|g\|_{p, \mu} + \|\Psi\|_{p, \mu} \|h\|_{p, \mu}}{1 - C_1 \|\Psi\|_{p, \mu}} = R_0.
\]

Finally, by using Shauder’s fixed point theorem, one concludes that (2.15) has a solution $x \in L^p([a, b])$. □

**Example 1:** Consider the following nonlinear Fredholm integral equation

\[
x(t) = g(t) + \lambda \int_0^1 \frac{\exp(5(s-t))}{5 \sqrt{t+s}} \left[ x(s) + \log(1 + x^2(s)) \right] ds, \quad t \in [0, 1], \tag{2.18}
\]

where $g \in L^2([0, 1]^2)$ and $\lambda > 0$ is real parameter. The reader can easily check that

\[
K(t, s) = \frac{\lambda \exp(5(s-t))}{5 \sqrt{t+s}} \in L^2([0, 1]^2).
\]
Moreover, numerical computations give us \( \|K\|_2 \approx 3.0030 \lambda \). On the other hand, the function \( f(s, x) = x + \log(1 + x^2) \) clearly satisfies the \( L^2 \)-Caratheodory’s condition. Moreover, since
\[
\lim_{x \to +\infty} \frac{|f(s, x)|}{x} = 1, \quad \forall s \in [0, 1],
\]
then \( \forall \epsilon > 0 \), there exists \( 1 < C_{1, \epsilon} < 1 + \epsilon, \ C_{2, \epsilon} > 0 \) such that
\[
|f(s, x)| \leq C_{1, \epsilon} |x| + C_{2, \epsilon}, \quad \forall x \in \mathbb{R}, \ s \in [0, 1]. \tag{2.19}
\]
Also, note that \( |f(s, x)| > x, \ \forall x > 0 \). Hence, it is necessary that \( C_{1, \epsilon} > 1 \). By using the previous inequality and theorem 2.11 one concludes that (2.18) has an \( L^2([0, 1]^2) \)-solution, whenever \( 0 \leq \lambda < \lambda_0 \approx 0.3330 \). If \( \lambda > \lambda_0 \), then theorem 2.11 is no longer applicable. Nonetheless, we use theorem 2.12 to prove the \( L^2 \)-existence result for larger values of \( \lambda \).

For this purpose, we let \( p = q = 2 \) and consider a weighted Lebesgue measure on \([0, 1], d\mu\), given by \( d\mu(t) = \mu(t) \, dt \), where \( \mu(t) = \exp(10t) \). Under the notation of theorem 6, the function \( \Psi(t) \) is given by the following formula,
\[
\Psi(t) = \left( \int_{0}^{1} \frac{|K(t, s)|^2}{\mu(s)} \, ds \right)^{1/2} = \frac{\lambda}{5} e^{5t} \sqrt{\log \left( \frac{t + 1}{t} \right)}. \tag{2.19}
\]
Note that \( \Psi \in L^2([0, 1], d\mu) \) and \( \|\Psi\|_{2, \mu} = \frac{\lambda}{5} \sqrt{2 \log(2)} \). Moreover, since the constant \( C_{1, \epsilon} \) in (2.19) satisfies \( 1 < C_{1, \epsilon} < 1 + \epsilon, \ \forall \epsilon > 0 \), then by using theorem 2.12 one concludes that (2.18) has an \( L^2([0, 1]^2) \)-solution, whenever \( 0 \leq \lambda < \lambda_1 = \frac{5}{\sqrt{2 \log(2)}} \approx 4.2466 \). This is a significant improvement of the result given by theorem 2.11.

We conclude this section by giving an existence and uniqueness result concerning nonlinear Volterra equation of the Hammerstein’s type.
2.3.1 Existence and uniqueness of continuous solution of a nonlinear Volterra equation

We first mention that the content of this paragraph has been borrowed entirely from [46]. Following is an existence result for Volterra integral equation of the Hammerstein’s type.

**Theorem 2.14** Consider the nonlinear Volterra integral equation

\[ x(t) = T x = g(t) + \int_a^t K(t, s) f(s, x(s)) \, ds, \quad t \in [a, b]. \]  

(2.20)

Assume that \( g \in C([a, b]) \). Also, assume that the kernel \( K \) satisfies the following conditions,

(i) \( K(t, s) \geq 0, \forall t, s \in [a, b], \quad K(t, s) \geq K(t_0, s), \forall t \leq t_0. \)

(ii) The function \( t \rightarrow \int_a^t K(t, s) \, ds \) is continuous on \([a, b]\).

(iii) \( \forall t_0 \in [a, b], \ s \rightarrow K(t_0, s) \in L^1([a, b]). \)

Moreover, assume that the function \( f(\cdot, \cdot) \) is continuous on \([a, b] \times \mathbb{R}\) and satisfies the following condition,

\[ |f(s, x)| \leq c_1 |x| + c_2, \quad \forall x \in \mathbb{R}, \]  

(2.21)

where \( c_1, c_2 \) are two positive constants. Also, assume that the kernel \( K \) satisfies the following condition

\[ \sup_{t \in [a, b]} \int_a^t K(t, s) \, ds < \frac{1}{c_1}. \]  

(2.22)

Then (2.20) has a solution \( x \in C([a, b]) \).

**Proof:** We first prove that the operator \( T \) associated with (2.20) is completely continuous from \( C([a, b]) \) into \( C([a, b]) \). To prove that \( T(C([a, b])) \subset C([a, b]) \), we proceed as
follows. Let \( x \in C([a, b]) \) and let \( t, t_0 \in [a, b], \) we may assume that \( t < t_0. \) Then, by using (i), one gets

\[
|Tx(t) - Tx(t_0)| \leq \int_a^t (K(t, s) - K(t_0, s)) |f(s, x(s))| \, ds + \int_t^{t_0} K(t_0, s) |f(s, x(s))| \, ds. 
\]  
(2.23)

By using (i), one gets

\[
0 \leq \int_a^t (K(t, s) - K(t_0, s)) \, ds \leq \left| \int_a^t K(t, s) \, ds - \int_a^{t_0} K(t, s) \, ds \right| + \int_t^{t_0} K(t_0, s) \, ds. 
\]  
(2.24)

By using (ii), (iii) and (2.24), one concludes that

\[
\lim_{t \to t_0} \int_t^{t_0} K(t_0, s) \, ds = \lim_{t \to t_0} \int_a^t (K(t, s) - K(t_0, s)) \, ds = 0. 
\]  
(2.25)

Since \( x \) is bounded on \([a, b]\) and since \( f(\cdot, \cdot) \) satisfies (2.21), then there exists a constant \( M > 0 \) such that

\[
|f(s, x(s))| \leq M, \quad \forall s \in [a, b]. 
\]  
(2.26)

By combining (2.23), (2.25) and (2.26), one concludes that \( Tx \in C([a, b]). \) Also, note that the continuity of \( T \) over \( C([a, b]) \) is a straightforward consequence of (ii) and the continuity of \( f(\cdot, \cdot) \) over \([a, b] \times \mathbb{R}. \) Next, to prove the compactness of \( T, \) it suffices to check that \( T \) satisfies the condition of Ascoli-Arzela’s theorem. Let \( S = \{x_n, n \in \mathbb{N}\} \) be a bounded set of \( C([a, b]) \) with a constant bound \( C_S. \) Then, \( \forall n \in \mathbb{N}, \) we have

\[
\|Tx_n\|_{\infty} \leq \|g\|_{\infty} + (c_1 C_S + c_2) \sup_{t \in [a, b]} \int_a^t K(t, s) \, ds = M_S. 
\]

Hence \( T(S) \) is uniformly bounded. Moreover, by substituting \( x \) by \( x_n \) in (2.23) and by using using (2.25) and (2.26), one shows that \( T(S) \) is equicontinuous. Hence, by using Ascoli-Arzela’s theorem, one concludes that \( T : C([a, b]) \to C([a, b]) \) is completely continuous. Next, let \( R > 0 \) be a positive real number and consider the closed, convex
ball of \( C([a, b]) \), denoted by \( B_R \) and given by \( B_R = \{ x \in C([a, b]), \|x\|_\infty \leq R \} \). Hence, one obtains the following inequality

\[ \|Tx\|_\infty \leq \|g\|_\infty + (c_1 R + c_2) \sup_{t \in [a,b]} \int_a^t K(t, s) \, ds. \]

Hence if \( \sup_{t \in [a,b]} \int_a^t K(t, s) \, ds < \frac{1}{c_1} \), then \( T(B_R) \subseteq B_R \), whenever

\[ R \geq \frac{\|g\|_\infty + c_2 \sup_{t \in [a,b]} \int_a^t K(t, s) \, ds}{1 - c_1 \sup_{t \in [a,b]} \int_a^t K(t, s) \, ds} = R_0. \]

By using Schauder’s fixed point theorem, one concludes that (2.20) has a continuous solution on \([a, b]\). \( \square \)

**Remark 2.3** Assume that in the previous theorem, the function \( f(\cdot, \cdot) \) satisfies the following inequality

\[ |f(s, x)| \leq c_1 |x|^\eta + c_2, \forall x \in \mathbb{R}, \forall s \in [a, b], \] (2.27)

where \( 0 \leq \eta < 1 \) and \( c_1, c_2 > 0 \). Then, there exist \( c_1', c_2' > 0 \) such that

\[ |f(s, x)| \leq c_1' |x|^\eta + c_2', \forall x \in \mathbb{R}, \forall s \in [a, b], \]

and such that \( \sup_{t \in [a,b]} \int_a^t K(t, s) \, ds < \frac{1}{c_1'} \). Hence, by the previous theorem, one concludes that if \( f(\cdot, \cdot) \) satisfies (2.27), then (2.20) has a solution \( x \in C([a, b]) \), no matter how large is \( b \).

**Remark 2.4** Since \( t \to \int_a^t K(t, s) \, ds \) is continuous, then condition (2.22) is satisfied for small enough \( b - a \). Hence, the previous theorem always ensures the existence of a solution of (2.20) in a neighborhood of \( a \).
Example 2 (Abel integral equation with power law nonlinearity)

Consider the following second kind Abel integral equation with power law nonlinearity,

\[ x(t) = g(t) + \int_0^t \frac{1}{(t-s)^\alpha} x(s)^\eta \, ds, \quad t \in [0, T], \quad (2.28) \]

where \( 0 \leq \alpha < 1 \), \( 0 \leq \eta \leq 1 \) and \( g \in C([0, T]) \). It is clear that the kernel

\[ K(t, s) = \frac{1}{(t-s)^\alpha} \chi_{[0,t]}(s) \]

satisfies conditions (i) and (iii) of theorem 2.14. Moreover, the function

\[ H(t) = \int_0^t K(t, s) \, ds = \frac{t^{1-\alpha}}{1-\alpha} \]

is continuous on \([0, T]\). If in (2.28), \( \eta \) is given by \( \eta = \frac{n}{2m+1} < 1 \), for some positive integers \( n, m \). Then, by using the previous theorem and Remark 2.3 one concludes that (2.28) has a continuous solution on \([0, T]\), for any real number \( T > 0 \). Finally, if \( \mu = 1 \), then the previous theorem ensures the existence of a continuous solution of (2.28) on \([0, T]\) for any positive real number \( T < (1 - \alpha)^{1/(1-\alpha)} \).

The uniqueness of the solution of (2.28) under a nonlinear condition, given by the following proposition.

Proposition 2.1 Assume that the function \( f(\cdot, \cdot) \) given by the previous theorem satisfies the following condition,

\[ |f(s, x) - f(s, y)| \leq L|x-y|^r, \quad \forall x, y \in \mathbb{R}, \quad (2.29) \]

for some constants \( L > 0 \) and \( 0 < r \leq 1 \). Then, under the conditions of the previous theorem, (2.20) has a unique continuous solution on \([a, b]\).
Hence, similarly, one gets

$$\|x - y\|_{\infty} \leq 0$$

Proof: Let $M_K$ be the positive constant given by $M_K = \sup_{t \in [a, b]} \int_a^t K(t, s) \, ds$. By contradiction, assume that (2.20) has two different solutions $x, y \in C([a, b])$, then there exists $0 \leq \epsilon < 1$ such that $\|x - y\|_{\infty} \geq \epsilon$. Moreover, it is clear that $\forall t \in [a, b]$, we have

$$|Tx(t) - Ty(t)| \leq L \int_a^t K(t, s)|x(s) - y(s)|^r \, ds \leq (\|x - y\|_{\infty})^r LM_K(t - a).$$

By using the previous inequality, one obtains

$$|T^2 x(t) - T^2 y(t)| \leq L \int_a^t K(t, s)|Tx(s) - Ty(s)|^r \, ds \leq (\|x - y\|_{\infty})^{r^2} (LM_K)^{r+1} \int_a^t (s - a)^r \, ds \leq (\|x - y\|_{\infty})^{r^2} (LM_K)^{r+1} \frac{(t - a)^{r+1}}{r + 1}.$$ 

Similarly, one gets

$$|T^3 x(t) - T^3 y(t)| \leq (\|x - y\|_{\infty})^{r^{2+1}} (LM_K)^{r^{2+1}} \frac{(t - a)^{r^{2+1}}}{(r + 1)(r^2 + r + 1)}.$$ 

more generally, for any positive integer $n$, we have

$$|T^n x(t) - T^n y(t)| \leq (\|x - y\|_{\infty})^{r^n} (LM_K)^{r^{n-1} + \cdots + r + 1} \frac{(t - a)^{r^{n-1} + \cdots + 1}}{(r + 1)(r^2 + r + 1) \cdots (r^{n-1} + \cdots + r + 1)}.$$ 

Hence,

$$\|T^n x - T^n y\|_{\infty} \leq (\|x - y\|_{\infty})^{r^n} \frac{(LM_K(b - a))^{r^{n-1} + \cdots + r + 1}}{(r + 1)(r^2 + r + 1) \cdots (r^{n-1} + \cdots + r + 1)} \leq \|x - y\|_{\infty} \frac{\|x - y\|_{\infty}^{r^{n-1} - 1} (LM_K(b - a))^{r^{n-1} + \cdots + r + 1}}{(r + 1)(r^2 + r + 1) \cdots (r^{n-1} + \cdots + r + 1)}.$$ 

Since $0 < r \leq 1$ and since $\|x - y\|_{\infty} \geq \epsilon$, for some $0 < \epsilon < 1$, then the previous inequality implies

$$\|T^n x - T^n y\|_{\infty} \leq \|x - y\|_{\infty} \frac{\max \left(1, (LM_K(b - a))^{1/(1-r)} \right)}{\epsilon (r + 1)(r^2 + r + 1) \cdots (r^{n-1} + \cdots + r + 1)}.$$ 

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Since $\epsilon \prod_{j=1}^{n} (r^j + \cdots + r + 1) \to +\infty$ as $n \to +\infty$, then there exists $N_0 \in \mathbb{N}$, such that

$$\max \left( 1, \frac{\left( LM_K(b-a)\right)^{1/(1-r)}}{\epsilon (r+1)(r^2+r+1) \cdots (r^{n-1} + \cdots + r + 1)} \right) < 1, \quad \forall n \geq N_0.$$ 

Consequently, we have

$$\|T^n x - T^n y\|_{\infty} < \|x - y\|_{\infty}, \quad \forall n \geq N_0. \quad (2.30)$$

On the other hand, since $x, y$ are solutions of (2.20), then they are fixed points of $T$ and more generally, they are fixed points of $T^n$, $\forall n \in \mathbb{N}$. Consequently, $x, y$ have to satisfy the equality, $\|T^n x - T^n y\|_{\infty} = \|x - y\|_{\infty}$, which contradicts (2.30). Hence, (2.20) has a unique solution. \(\Box\)

2.3.2 $L^p$–existence solutions of nonlinear integral equations of the Hammerstein and Urysohn types on unbounded intervals.

In this section, we consider a positive valued function $\mu(\cdot)$ which is continuous on $\mathbb{R}^+$ and satisfies the condition that there exists $a \geq 0$ such that the function

$$\delta \to \sup_{t \geq a} \frac{\mu(t)}{\mu(t + \delta)}, \quad \delta > 0 \quad \text{is bounded in a neighborhood of zero} \quad (2.31)$$

Our existence result for the Hammerstein’s integral equation (2.4) is given by the following theorem.

**Theorem 2.15** Consider the following nonlinear Hammerstein’s integral equation

$$x(t) = g(t) + \int_{0}^{+\infty} k(t,s)f(s,x(s))ds, \quad (2.32)$$
Let \( p \geq 1, q \geq 1 \) be two positive real numbers satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \). Under the above notation and assumptions, let \( A \geq a \) and define the function \( \varphi_A(\cdot) \) by

\[
\varphi_A(s) = \int_A^{+\infty} |k(t,s)|\mu(t)\,dt. \tag{2.33}
\]

Assume that \( g(\cdot) \in L_{p,\mu} \), \( k(\cdot, \cdot) \) is bounded and continuous on \( \mathbb{R}^2_+ \). Moreover, assume that

(i) the function \( (t, x) \rightarrow f(t, x) \) is of Caratheodory’s type.

(ii) There exist \( c > 0 \) and \( h \in L_{p,\mu} \) such that \( |f(s, x)| \leq h(s) + c|x| \) a.e. \( s \geq 0 \).

(iii) \( \lim_{A \to +\infty} \|\varphi_A\|_\infty = 0 \).

(iv) \( \int_{\mathbb{R}_+} |k(t, s)|\mu(t)\,dt = \varphi_0(s) \leq \alpha_1 \) a.e. \( s \geq 0 \).

(v) \( \int_{\mathbb{R}_+} k(t, s)[\mu(s)]^{-\frac{p}{q}}\,dt \leq \alpha_2 \) a.e \( t \geq 0 \).

(vi) \( c\alpha_1^{\frac{1}{p}} \alpha_2^{\frac{1}{q}} < 1 \),

Under the above assumptions, the Hammerstein’s integral equation (2.32) has at least one solution \( x \in L_{p,\mu} \).

**Proof:** Let \( T \) be the operator defined on \( L_{p,\mu} \) by

\[
Tx(t) = g(t) + \int_{\mathbb{R}_+} k(t, s)f(s, x(s))\,ds. \tag{2.34}
\]

The proof of the theorem is done as follows. We first prove that there exists a closed ball \( B_R \) in \( L_{p,\mu} \) such that \( T(B_R) \subset B_R \). Let’s first check that \( T(L_{p,\mu}) \subset L_{p,\mu} \). By using (ii), one gets

\[
\|Tx\|_{p,\mu} \leq \|g\|_{p,\mu} + \left[ \int_{\mathbb{R}_+} \left[ \int_{\mathbb{R}_+} |k(t, s)||c|x(s)| + h(s)|ds\right]^p \mu(t)\,dt \right]^{1/p}
\]

\[
\leq \|g\|_{p,\mu} + \left[ \int_{\mathbb{R}_+} \left[ \int_{\mathbb{R}_+} |k(t, s)|^{\frac{1}{q}}[\mu(s)]^{-\frac{1}{q}}[|k(t, s)||\mu(s)||^\frac{1}{q}|c|x(s)| + h(s)|ds\]^{p} \mu(t)\,dt \right]^{1/p}
\]
Since $k(\cdot, \cdot)$ is bounded, then the function $[|k(t, s)||\mu(s)|]^{\frac{1}{p}}|c|x(s)| + h(s)$ belongs to $L_p$, then by combining Hölder’s inequality and Fubini’s theorem, one obtains the following inequalities

$$\|Tx\|_{p,\mu} \leq \|g\|_{p,\mu} + \frac{1}{2} \left[ \left( \int_{\mathbb{R}^+} (c|x(s)| + h(s))^p \left( \int_{\mathbb{R}^+} |k(t, s)||\mu(t)|dt \right) \mu(s)ds \right]^{1/p} \right.$$

$$\leq \|g\|_{p,\mu} + \frac{1}{2} \alpha_1^{1/q} \alpha_2^{1/p} \left[ c\|x\|_{p,\mu} + \|h\|_{p,\mu} \right].$$

Hence, the operator $T$ maps $L_{p,\mu}$ into itself. Moreover, it is clear that if $R \geq R_0 = \frac{\|g\|_{p,\mu} + \alpha_1^{1/q} \alpha_2^{1/p} \|h\|_{p,\mu}}{1 - c \alpha_1^{\frac{1}{p}} \alpha_2^2}$, then $T(B_R) \subset B_R$, where $B_R$ is the closed ball of $L_{p,\mu}$ with center 0 and radius $R$.

To prove that $T$ has a fixed point in $L_{p,\mu}$, we use Schauder’s fixed point theorem. Hence, we need to prove that the operator $T$ is continuous and compact.

To prove that $T$ is compact, we first consider a uniformly bounded set $X$ of $L_{p,\mu}$ (i.e there exists $M > 0$ such that $\|x\|_{p,\mu} \leq M$, $\forall x \in X$). Let $A > 0$, then straightforward computations show that

$$\left[ \int_{\mathbb{R}^+}^- \inf_{t \geq A} |Tx(t)|^p \mu(t)dt \right]^{1/p} \leq \left[ \int_{\mathbb{R}^+}^- |g(t)|^p \mu(t)dt \right]^{1/p}$$

$$+ \alpha_2^{1/q} \left[ \int_{\mathbb{R}^+}^- \mu(s)(c|x(s)| + h(s))^p \left( \int_{\mathbb{R}^+}^- \mu(t)|k(t, s)|dt \right) ds \right]^{1/p}$$

$$\leq \alpha_2^{1/q} \|\varphi_A\|_{\infty} (c\|x\|_{p,\mu} + \|h\|_{p,\mu})^p + \left[ \int_{\mathbb{R}^+}^- |g(t)|^p \mu(t)dt \right]^{1/p}.$$

Since

$$\lim_{A \to +\infty} \|\varphi_A\|_{\infty} = 0 \quad \text{and} \quad t \mapsto |g(t)|^p \mu(t) \in L^1[0, +\infty],$$

then

$$\lim_{A \to +\infty} \int_{\mathbb{R}^+}^- |Tx(t)|^p \mu(t)dt = 0 \quad (2.35)$$
Now, let \( \delta > 0 \) and let \( T_kx(t) = \int_0^{+\infty} k(t,s)f(s,x(s)) \, ds \), then

\[
\left[ \int_0^{+\infty} \left| T_kx(t+\delta) - T_kx(t) \right|^p \mu(t) \, dt \right]^{1/p} \\
\leq \left[ \int_0^{+\infty} \left| g(t+\delta) - g(t) + T_kx(t+\delta) - T_kx(t) \right|^p \mu(t) \, dt \right]^{1/p} \\
\leq \| g(t+\delta) - g(t) \|_{p,\mu} + \| T_kx(t+\delta) - T_kx(t) \|_{p,\mu}.
\]

Since \( g(\cdot) \in L_{p,\mu} \), then

\[
\lim_{\delta \to 0} \| g(t+\delta) - g(t) \|_{p,\mu}^p = 0 \tag{2.36}
\]

Next, it is easy to see that \( \forall A \geq a \), where \( a \) is as given by \( \text{(2.31)} \), we have

\[
\| T_kx(t+\delta) - T_kx(t) \|_{p,\mu}^p \leq \int_0^{+\infty} \left| \int_0^{+\infty} (k(t+\delta,s) - k(t,s))f(s,x(s)) \right|^p \mu(t) \, dt
\]

\[
\leq 2\alpha^\frac{p}{q} \int_0^{+\infty} \left( (h(s) + c|x(s)|)^p \mu(s) \right) \left( \int_0^{+\infty} |k(t+\delta,s) - k(t,s)| \mu(t) \, dt \right) \, ds
\]

\[
\leq 2\alpha^\frac{p}{q} \int_{s \geq 0} \left( (h(s) + c|x(s)|)^p \mu(s) \right) \left( \int_{t \geq A} |k(t+\delta,s) - k(t,s)| \mu(t) \, dt \right) \, ds
\]

\[
+ 2\alpha^\frac{p}{q} \int_{s \geq A} \left( (h(s) + c|x(s)|)^p \mu(s) \right) \left( \int_{0 \leq t \leq A} |k(t+\delta,s) - k(t,s)| \mu(t) \, dt \right) \, ds
\]

\[
+ 2\alpha^\frac{p}{q} \int_{0 \leq s \leq A} \left( (h(s) + c|x(s)|)^p \mu(s) \right) \left( \int_{0 \leq t \leq A} |k(t+\delta,s) - k(t,s)| \mu(t) \, dt \right) \, ds.
\]

Since

\[
\int_{t \geq A} |k(t+\delta,s) - k(t,s)| \mu(t) \, dt \leq \int_{t \geq A} |k(t+\delta,s)| \mu(t) \, dt + \int_{t \geq A} |k(t,s)| \mu(t) \, dt
\]

\[
\leq C_\delta \| \varphi_{A+\delta} \|_\infty + \| \varphi_A \|_\infty,
\]

\( C_\delta = \sup_{t \geq A} \left( \frac{\mu(t)}{\mu(t+\delta)} \right) \),

then,

\[
\lim_{A \to +\infty} \int_{t \geq A} |k(t+\delta,s) - k(t,s)| \mu(t) \, dt = 0 \tag{2.37}
\]

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Also, since \( X \) is a uniformly bounded subset of \( L_{p,\mu} \), then by using the previous techniques, one can easily check that

\[
\lim_{A \to +\infty} 2\alpha_2^p \int_{s \geq 0} (h(s) + c|x(s)|)^p \mu(s) \left( \int_{t \geq A} |k(t + \delta, s) - k(t, s)| \mu(t) dt \right) ds = 0 \tag{2.38}
\]

Finally, the continuity of \((t, s) \to \mu(t)k(t, s)\) on \([0, A]^2\) gives us

\[
\lim_{\delta \to 0} \int_{0 \leq t \leq A} |k(t + \delta, s) - k(t, s)| \mu(t) dt = 0 \tag{2.39}
\]

Collecting everything together, one concludes that \( T : B_R \to B_R \) is compact.

We should mention that the proof of the continuity of \( T \) over \( L_{p,\mu} \) can be done by using the same techniques as in the continuity proof of the Urysohn’s operator given in theorem 2.16 of the next section and is omitted. Finally, by using the above results as well as Schauder’s fixed point theorem, one concludes that the integral operator \( T \) has a fixed point. This concludes the proof of the theorem. \( \square \)

**Example 1:** Let \( \mu(s) = e^{-s} \), \( p = q = 2 \) and consider the nonlinear Hammerstein’s integral equation

\[
x(t) = g(t) + \lambda \int_0^{+\infty} \frac{te^{-s}}{1 + (t + s)^2} \left( x(s) + \frac{1}{(1 + s)(1 + x(s)^2)} \right) ds. \tag{2.40}
\]

In this case, we have

\[
f(s, x) = x + \frac{1}{(1 + s)(1 + x^2)} \quad c = 1, \quad h(s) = \frac{1}{1 + s} \in L_{2,\mu}.
\]

Moreover, by using the notation of the previous theorem, it is easy to see that

\[
\varphi_A(s) = e^{-s} \int_A^{+\infty} \frac{te^{-t}}{1 + (t + s)^2} dt \leq e^{-s} \frac{1}{1 + A^2} \to 0,
\]

uniformly in $s$. On the other hand, we have
\[
\int_0^{+\infty} |k(t, s)|\mu(t)\, dt = e^{-s} \int_0^{+\infty} \frac{te^{-t}}{1 + (t + s)^2}\, dt \leq \alpha_1 \approx 0.3434, \quad \forall s \geq 0.
\]
Moreover, since
\[
\int_0^{+\infty} |k(t, s)|\mu(s)^{-p/q}\, ds = \int_0^{+\infty} \frac{t}{1 + (t + s)^2}\, ds \leq \alpha_2 = 1,
\]
then, the previous theorem implies that (2.40) has a solution in $L_{2,\mu}$, whenever $|\lambda| \leq 2.9122$.

We conclude this section by the extension of the above existence result to the nonlinear integral equations of the Urysohn’s case. This extension is borrowed from [47]. It is given as follows. Consider a positive $L^1$-function $\mu(\cdot)$ on $\mathbb{R}_+$. Then the following theorem provides us with a fairly general $L_{p,\mu}$-existence result for nonlinear Urysohn’s integral equations
\[
x(t) = g(t) + \int_0^{+\infty} f(t, s, x(s))\, ds.
\](2.41)
Here $g \in L_{p,\mu}$, and $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory’s condition. Note that this theorem has the advantage of handling polynomial types of nonlinearity over an unbounded domain.

**Theorem 2.16** Under the above notation and assumptions, assume that there exists a function $k(t, s)$ continuous on $\mathbb{R}_+^2$, a real number $r \geq 1$, and $h \in L_{r,\mu}$, such that
\begin{enumerate}[(i)]
\item $|f(t, s, x)| \leq k(t, s) \left( h(s) + |x|^{p/r} \right)$, a.e. $t, s \geq 0$ and $\forall x \in \mathbb{R}$,
\item $\int_{\mathbb{R}_+} \mu(t) |\psi(t)|^p\, dt < +\infty$, $\frac{1}{r} + \frac{1}{r'} = 1$, where
\[
\psi(t) = \left[ \int_{\mathbb{R}_+} |k(t, s)|^{r'} (\mu(s))^{-r/r'}\, ds \right]^{1/r'}, \quad \forall t \geq 0.
\]
\end{enumerate}
(iii) There exists $R > 0$ such that $\|g\|_{p, \mu} + (\|h\|_{r, \mu} + R^{p/r}) \cdot \|\psi\|_{p, \mu} \leq R$.

(iv) There exists a function $L_f(\cdot)$ continuous in a neighborhood of 0 with $L_f(0) = 0$, $v \in L^1(\mathbb{R}_+)$, and $b \in L^r(\mathbb{R}_+)$ such that

\[ |f(t, s, x) - f(\tau, s, x)| \leq L_f(|t - \tau|)(v(s) + b(s)|x|^{p/r}), \quad \text{a.e. } t, \tau, s \geq 0 \text{ and } \forall x \in \mathbb{R}. \]

Then, (2.41) has a solution in $L_{p, \mu}$.

Proof: see [47].
Chapter 3

$L^p$-Existence solutions of nonlinear quadratic integral equations defined on unbounded intervals

The main subject of this chapter is the study of existence of solutions of the following nonlinear quadratic integral equation:

\[ x(t) = a(t) + x(t) \int_0^{+\infty} k(t, s) h(s, x(s)) ds, \quad t \geq 0 \]  \hspace{1cm} (3.1)

where the unknown $x$ and $a(.)$ are in $L^p(\mathbb{R}_+)$. Also, in this chapter, we give an existence result by means of Schayder’s fixed pint theorem of the following more general quadratic equation

\[ x(t) = a(t) + f(t, x(t)) \int_0^{+\infty} u(t, s, x(s)) ds, \quad t \geq 0. \]  \hspace{1cm} (3.2)

The above equations appears very often, in a lot of applications to real world problems. For example, some problems considered in vehicular traffic theory, biology and queuing theory lead to the quadratic integral equations of this type (cf. [5, 22, 29, 73]). Moreover, such integral equations are also applied in the theory of radiative transfer and the theory
of neutron transport as well in the kinetic theory of gases (cf. [19] [22], among others). Thus a quadratic integral equation of the form (3.2) creates a generalization of several kinds of quadratic integral equations of the above-mentioned type. On the other hand, the problems of the existence of solutions of Eq. (3.1) have received very little attention in the literature to the best of our knowledge. Only for the case $x$ is continuous on $[a, b]$, continuous and bounded on $\mathbb{R}_+$ or $L^1([0, 1])$, the existence of solutions has been obtained by some authors (see [8] [14] [27] [57] [65], among others), such that $h(., x(\cdot))$ satisfying the Lipchitz condition in the variable $x$ or $|h(., x(\cdot))| \leq a(\cdot) + b|x(\cdot)|$, when $b > 0$ and $a(\cdot)$ are given.

Investigations of solutions use mostly the Banach fixed-point principle, the Schauder fixed-point theorem, and successive approximations (see [49]). The considerations of this chapter are based on the notion of a measure of non-compactness in Banach space and the fixed-point theorem of Darbo type (Banas and Goebel [12] and Darbo [26]). This approach allows us to weaken the conditions for the existence of solutions of a quadratic integral equation.

In the following section we collect definitions and results concerning the concept of measure of noncompactness and related fixed point theorems.

### 3.1 Measure of noncompactness in Banach space and fixed point theorems.

#### 3.1.1 Preliminaries on measures of noncompactness

There are many generalizations of the Schauder Fixed Point Theorem. We mention here one which shows that the assumption of compactness of the operator can be relaxed.
To this purpose we need a tool which will measure how much noncompact the operator actually is. We refer the reader to see [12, 29].

Let \( \mathfrak{M} \) be the family of all bounded subsets of the Banach space \( X \). Analogously denote by \( \mathfrak{N} \) the family of all relatively compact and nonempty subsets of \( X \). recall that \( B \subset X \) is said to be bounded if \( B \) is contained in some ball. If \( B \subset \mathfrak{M} \) is not relatively compact (precompact) then there exists an \( \epsilon > 0 \) such that \( B \) cannot be covered by a finite number of \( \epsilon \)-balls, and it is then also impossible to cover \( B \) by finitely many sets of diameter \( < \epsilon \); recall that

\[
\text{diam}(B) = \sup_{x,y \in B} \{|x - y|\}
\]

is recalled the diameter of \( B \).

**Definition 3.1** Let \( X \) be a Banach space and \( \mathfrak{M} \) its bounded sets. Then \( \mu : \mathfrak{M} \to \mathbb{R}_+ \), defined by

\[
\mu(B) = \inf \{ r > 0 : B \text{ can be expressed as the union } B = \bigcup_{i=1}^{n} B_i \text{ of a finite number of sets } B_i \text{ with diameter } < r \},
\]

is called the (Kuratowski-)measure of noncompactness.

**Proposition 3.1** (Properties of the Kuratowski measure of noncompactness). Let \( X \) be a (real or complex) Banach space. Then for all bounded subsets \( M, M_1, ..., M_n, N \) of \( X \) the following assertions hold:

(i) \( \mu(\emptyset) = 0 \),

(ii) \( \mu(M) = 0 \iff M \in \mathfrak{N} \),

(iii) \( 0 \leq \mu(M) \leq \text{diam}(M) \),

(iv) \( M \subset N \Rightarrow \mu(M) \leq \mu(N) \),

(v) \( \mu(M + N) \leq \mu(M) + \mu(N) \),
(vi) $\mu(\lambda M) \leq |\lambda|\mu(M)$ for all $\lambda \in \mathbb{R}$ (or $\mathbb{C}$),

(vii) $\mu(M) \leq \mu(M)$,

(viii) $\mu(\bigcup_{j=1}^{n} M_j) = \max\{\mu(M_1), ..., \mu(M_n)\}$,

(ix) $\mu(M) = \mu(\text{conv}(M))$.

Recall that a set $M$ in complete metric space is relatively compact if and only if it is totally bounded (iff $\mu(M) = 0$), and also that a decreasing sequence of nonempty compact sets is nonempty, these facts give the foundations and flavor of the next proposition.

**Proposition 3.2** Let $(X,d)$ be a complete metric space and $A_1 \supset A_2 \supset ........$ be a decreasing sequence of nonempty, closed subsets of $X$. call $A_\infty = \bigcap_{n \geq 1} A_n$.

If $\lim_{n \to \infty} \mu(A_n) = 0$ then $A_\infty$ is a nonempty compact set.

Notice that, throughout this thesis, we use the following definition given by J. Banas and I. Goebel [12] to the measure of noncompactness in Banach space.

**Definition 3.2** A function $\mu : \mathcal{M}_p \longrightarrow \mathbb{R}_+$ will be called a measure of noncompactness if it satisfies to the following conditions:

1- $\ker \mu(X) = \{X \in \mathcal{M}_p : \mu(X) = 0\}$ is nonempty and $\ker \mu(X) \subset \mathcal{N}_p$.

2- $X \subset Y \implies \mu(X) \leq \mu(Y)$.

3- $\mu(X) = \mu(X)$.

4- $\mu(\text{conv}X) = \mu(X)$.

5- $\forall 0 \leq \lambda \leq 1, \quad \mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y)$.
6- If \( \{X_n\}_{n \geq 1} \) is a sequence of closed sets from \( \mathcal{M}_p \) such that

\[
X_{n+1} \subset X_n, \quad n = 1, 2, \ldots
\]

and

\[
\lim_{n \to \infty} \mu(X_n) = 0.
\]

Then the intersection set \( X_\infty = \cap_{n>0} X_n \) is nonempty.

### 3.1.2 Measure of noncompactness in \( L^p(\mathbb{R}_+) \)

To introduce the notion of measure of noncompactness in \( L^p(\mathbb{R}_+) \), we let \( \mathcal{M}_p \) (resp. \( \mathcal{N}_p \)) denote the family of all nonempty and bounded (resp. nonempty and relatively compact) subsets of \( L^p(\mathbb{R}_+) \). We will adopt the following definition of measure of noncompactness [14].

**Definition 3.3** A function \( \mu : \mathcal{M}_p \to \mathbb{R}_+ \) will be called a measure of noncompactness if it satisfies to the following conditions:

1- \( \ker \mu(X) = \{X \in \mathcal{M}_p : \mu(X) = 0\} \) is nonempty and \( \ker \mu(X) \subset \mathcal{N}_p \).

2- \( X \subset Y \implies \mu(X) \leq \mu(Y) \).

3- \( \mu(\overline{X}) = \mu(X) \).

4- \( \mu(\text{conv}X) = \mu(X) \).

5- \( \forall 0 \leq \lambda \leq 1, \quad \mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y) \).

6- If \( \{X_n\}_{n \geq 1} \) is a sequence of closed sets from \( \mathcal{M}_p \) such that

\[
X_{n+1} \subset X_n, \quad n = 1, 2, \ldots
\]
and
\[
\lim_{n \to \infty} \mu(X_n) = 0.
\]

Then the intersection set \( X_\infty = \cap_{n>0} X_n \) is nonempty.

In particular, the measure of noncompactness in \( L^p(\mathbb{R}_+) \) is defined as follows. Let \( X \) be a fixed nonempty and bounded subset of \( L^p(\mathbb{R}_+) \). For \( x \in X \), denote by
\[
w(X) = \lim_{\delta \to 0} \sup \left\{ \left( \int_0^{+\infty} |x(t + \delta) - x(t)|^p dt \right)^{1/p}, \ x \in X \right\},
\]
and
\[
\nu(X) = \lim_{T \to +\infty} \sup \left\{ \left( \int_T^{+\infty} |x(t)|^p dt \right)^{1/p}, \ x \in X \right\}.
\]
Finally, for any subset \( X \) of \( \mathcal{M}_p \), let us denote
\[
\mu(X) = w(X) + \nu(X).
\]
It can be easily shown, that \( \mu \) is a measure of noncompactness in \( L^p(\mathbb{R}_+) \).

### 3.1.3 Darbo’s fixed point principle

In the next definition we will consider a special class of continuous and bounded operators.

**Definition 3.4** Let \( T : M \subset X \to X \) be a bounded operator from a Banach space \( X \) into itself. The operator \( T \) is called a \( k \)-set contraction if there is a number \( k \geq 0 \) such that
\[
\mu(T(A)) \leq k \mu(A)
\]
for all bounded sets \( A \) in \( M \). The bounded operator \( T \) is called condensing if \( \mu(T(A)) < \mu(A) \) for all bounded sets \( A \) in \( M \) with \( \mu(M) > 0 \).
Obviously, every $k$-set contraction for $0 \leq k < 1$ is condensing. Every compact map $T$ is a $k$-set contraction with $k = 0$. A typical example of a $k$-set contraction with $0 \leq k < 1$ is the following one.

**Example:** Let $K, C : D \subset X \to X$ be operators on a Banach space $X$. Let $K$ be a $k$-contractive, i.e., there exists $k \in [0, 1)$ such that

$$\|K(x) - K(y)\| \leq k\|x - y\|$$

for all $x, y \in D$, and $C$ be compact. Then $K + C$ is a $k$-set contraction. Indeed, let $A \subset D$ be a bounded set. It follows from Definition that $\mu(K(A)) \leq k\mu(A)$. By (ii) of Proposition 3.1 we have $\mu(C(A)) = 0$. Set $T := K + C$. Now (iv) and (v) of Proposition 3.1 imply

$$\mu(T(A)) \leq \mu(K(A) + C(A)) \leq \mu(K(A)) + \mu(C(A)) \leq k\mu(A).$$

The following assertion is a generalization of the Schauder Fixed Point Theorem (note that every compact operator is condensing).

**Theorem 3.1 (Darbo’s fixed point theorem).** Let us suppose that

(i) $M$ is a nonempty, closed, bounded and convex subset of a Banach space $X$;

(ii) an operator $T : M \subset X \to M$ is condensing and continuous on $M$.

Then $T$ has a fixed point in $M$.

Proof. The idea of the proof is to find a suitable subset $A$ of $M$ which is mapped into itself by $T$ in such a way that the Schauder Fixed Point Theorem can be applied to the restriction $T : A \to A$. The resulting fixed point is then trivially a fixed point of the original mapping $T : M \to M$. The set $A$ is constructed in the following way. Choose
a point \( m \in M \) and let \( \Lambda \) denote the system of all closed, convex subsets \( K \) of \( M \) for which \( m \in K \) and \( T(K) \subset K \). Set

\[
\bigcap_{K \in \Lambda} K \quad \text{and} \quad C = \overline{\text{conv}(T(A) \cup \{m\})}.
\]

Since \( m \subset A \) and \( T(A) \subset A \), it follows that \( C \subset A \). This implies \( T(C) \subset T(A) \).

Obviously \( T(A) \subset C \), i.e., \( T(C) \subset C \) which means that \( C \subset \Lambda \). So, \( A \subset C \). We have proved that \( A = C \). Now, (vii), (viii) and (ix) of Proposition 3.1 imply that

\[
\mu(A) = \mu(C) = \mu(T(A)).
\]

Since \( T \) is condensing,

\[
\mu(A) = 0.
\]

Since \( A \) is also closed, \( A \) is a compact set. The restriction of \( T \) to \( A \) is thus a compact operator. Consequently, the Schauder Fixed Point Theorem can be applied to the mapping \( T : A \to A \).

### 3.2 \( L^p \)—existence results for nonlinear quadratic equations

In this paragraph, we study the existence of a solution of the following nonlinear integral equation

\[
x(t) = a(t) + x(t) \int_0^{+\infty} k(t, s)h(s, x(s))ds, \quad t \geq 0
\]

by the use of the concept of measure of non compactness in \( L^p \) and Darbo’s fixed point theorem. This is the subject of the following theorem.

**Theorem 3.2** Consider the nonlinear quadratic integral equation (3.6). Assume that:
(H1) \((s, x) \mapsto h(s, x)\) is a function from \(\mathbb{R}_+ \times \mathbb{R}\) into \(\mathbb{R}\) satisfying the Caratheodory’s condition.

(H2) There exist \(b \in L^{r'}(\mathbb{R}_+)\) and \(c > 0\) such that \(|h(s, x(s))| \leq b(s) + c|x(s)|^{p/r}\).

(H3) \((t, s) \mapsto k(t, s)\) is a function from \(\mathbb{R}_+ \times \mathbb{R}\) into \(\mathbb{R}\) such that \(k(t, \cdot)\) belongs \(L^{r'}(\mathbb{R}_+)\)

\((1/r + 1/r' = 1)\) for a.e \(t \geq 0\).

(H4) The function \(\Pi : t \mapsto k_t = \|k(t, \cdot)\|_{r'} \in L^\infty(\mathbb{R}_+)\), there exists \(M_0 > 0\) and a function \(\Delta\) continuous on a neighborhood of 0 such that \(\|k(t, \cdot)\|_{r'} \leq M_0\) and

\[\|\Pi(\cdot + \delta) - \Pi(\cdot)\|_\infty \leq \Delta(\delta)\text{ with } \Delta(0) = 0.\]

- Moreover, let \(\Psi\) be the function defined on \(\mathbb{R}_+\) by \(\Psi(R) = \|b\|_r + cR^{p/r}\), and assume that there exists a positive solution \(R_0\) of the inequality

\[\|a\|_p + RM_0\Psi(R) \leq R,\]

with \(M_0\Psi(R_0) < 1\).

Under the above assumptions, equation (3.6) has at least one solution which belongs to the space \(L^p(\mathbb{R}_+)\).

Proof. Let \(H\) be the operator defined by

\[Hx(t) = a(t) + x(t) \int_0^{+\infty} k(t, s)h(s, x(s))ds, \quad t \geq 0.\]

Under the assumptions (H1) - (H4), we have for \(x \in L^p(\mathbb{R}_+)\),

\[|Hx(t)| \leq |a(t)| + |x(t)| \int_0^{+\infty} |k(t, s)| \left[ b(s) + c|x(s)|^{p/r} \right] ds.\]
Then, Hölder’s inequality implies

$$\|Hx\|_p \leq \|a\|_p + M_0 \Psi(\|x\|_p) \|x\|_p < +\infty$$

In what follows we show that $T$ is continuous on $L^p(\mathbb{R}_+)$. Let $F$ be the operator defined by

$$Fx(t) = x(t) \int_0^{+\infty} k(t, s) h(s, x(s)) ds.$$  

Let $x, x_n$ in $L^p(\mathbb{R}_+)$ be such that

$$\lim_{n \to +\infty} \|x_n - x\|_p = 0.$$  

Suppose that $\|Fx_n - Fx\|_p$ does not converge to 0 as $n \to \infty$, thus there exist $\epsilon > 0$ and a subsequence $x_{n_j}$ of $x_n$ such that

$$\|Fx_{n_j} - Fx\|_p > \epsilon, \quad \forall j = 1, 2, ... \tag{3.7}$$

and $x_{n_j}(t)$ converge to $x(t)$ as $n \to \infty$ for a.e $t \geq 0$. First, from the inequality

$$\|x_n\|_p \leq \|x_n - x\|_p + \|x\|_p$$

it follows that $\|x_n\|_p$ is bounded. First, one has

$$\int_0^{+\infty} k(t, s) h(s, x(s)) ds \leq \|k(t, .)\|_r(\|b\|_r + c\|x_{n_j}\|_p) \in L^1(\mathbb{R}_+).$$

The Lebesgue dominated convergence theorem and the continuity of $f$ implies that

$$\lim_{n_j \to \infty} (Fx_{n_j})(t) = (Fx)(t).$$

In the other hand, for any subset $A$ of $\mathbb{R}_+$ one has,

$$\int_A |Fx_{n_j}(t)|^p dt \leq \|k_t x_A\|_\infty \left(\|b\|_r + c\|x_{n_j}\|_p/\epsilon\right) \int_A |x_{n_j}(t)|^p dt.$$
where $\chi_A$ denotes the characteristic function on the subset $A \subset \mathbb{R}_+$. The Vitali convergence theorem implies that

$$\lim_{n \to +\infty} \|F x_{n_j} - F x\|_p = 0,$$

which contradicts (3.7). Hence, the operator $F$ is continuous on $L^p(\mathbb{R}_+)$ and consequently, $U$ is also continuous on $L^p(\mathbb{R}_+)$. \hfill \square

Moreover, in view of the assumption (H4), the operator $H$ transforms the ball $B_{R_0}$ into itself.

Now, let us fix a nonempty subset $X$ of $B_{R_0}$. We first consider a real number $A > 0$ and an arbitrary fixed $x$ in $X$. Then we have

$$\left( \int_{A}^{+\infty} |Hx(t)|^p dt \right)^{1/p} = \left( \int_{A}^{+\infty} \left| a(t) + x(t) \int_{0}^{+\infty} k(t, s)h(s, x(s))ds \right|^p dt \right)^{1/p} \leq \left( \int_{A}^{+\infty} |a(t)|^p dt \right)^{1/p} + \left( \int_{A}^{+\infty} |x(t)|^p \left\| k(t, \cdot) \right\|_p \left( \left\| b \right\|_r + c \left\| x \right\|^{p/r}_r \right) dt \right)^{1/p} \leq \left( \int_{A}^{+\infty} |a(t)|^p dt \right)^{1/p} + M_0 \Psi(R_0) \left( \int_{A}^{+\infty} |x(t)|^p dt \right)^{1/p}. $$

Keeping in mind that

$$\lim_{A \to +\infty} \left( \int_{A}^{+\infty} |a(t)|^p dt \right)^{1/p} = 0,$$

we obtain

$$\nu(HX) \leq M_0 \Psi(R_0) \nu(X).$$

Here, $\nu(\cdot)$ is as given by (3.4). On the other hand, consider a real number $\delta > 0$ and an arbitrary fixed $x$ in $X$, then we have

$$|Hx(t+\delta) - Hx(t)| \leq |a(t+\delta) - a(t)| + |x(t+\delta) - x(t)| \leq |a(t+\delta) - a(t)| + \left| \int_{0}^{+\infty} k(t+\delta, s)h(s, x(s))ds \right| + \left| \int_{0}^{+\infty} k(t, s)h(s, x(s))ds \right| \leq |a(t+\delta) - a(t)| + \left| \int_{0}^{+\infty} |k(t+\delta, s)|h(s, x(s))ds \right| + \left| \int_{0}^{+\infty} |k(t, s)|h(s, x(s))ds \right|.$$
Hence, we have
\[
\|Hx(. + \delta) - Hx(.)\|_p \leq \|a(. + \delta) - a(.)\|_p + M_0 \Psi(R_0)\|x(. + \delta) - x(.)\|_p + R_0 \Psi(R_0)\|\Pi(. + \delta) - \Pi(.)\|_\infty.
\]

Taking into account that
\[
\lim_{\delta \to 0} \|a(. + \delta) - a(.)\|_p = 0 \quad \text{and} \quad \lim_{\delta \to 0} \|\Pi(. + \delta) - \Pi(.)\|_\infty \leq \lim_{\delta \to 0} \Delta(\delta) = 0,
\]
we get
\[
w(HX) \leq M_0 \Psi(R_0)w(X),
\]
where, \(w(.)\) is given by (3.3). Consequently, we arrive at the following inequality:
\[
\mu(HX) \leq M_0 \Psi(R_0)\mu(X).
\]

Here, \(\mu(.)\) is the measure of noncompactness in \(L^p(\mathbb{R}_+)\), given by (3.5). This means that the operator \(H\) is a contraction with respect to \(\mu\). Finally, since \(M_0 \Psi(R_0) < 1\), then by applying Theorem 3.1, we conclude that equation (3.6) has at least one solution belonging to the space \(L^p(\mathbb{R}_+)\). \(\Box\)

**Example:** Let \(T\) be a real positive number and consider (3.6) with the special cases \(a(t) = \chi_{[0,T]}(t)\), \(u(t, s, x) = \frac{t}{t+s} \varphi(s)x\chi_{[0,T]}(s)\). Thus, we obtain the following nonlinear quadratic integral equation
\[
x(t) = 1 + x(t) \int_0^T \frac{t}{t+s} \varphi(s)x(s)ds, \quad \text{a.e } t \in [0, T]
\]
Where $\chi_A$ denotes the characteristic function of a subset $A \subset \mathbb{R}$. The above equation is the famous quadratic integral equation of Chandrasekhar type and was considered in many papers, see for example [34]. If $T \in ]0,1[$, then we prove that the equation

$$x(t) = 1 + x(t) \int_0^T \frac{t \sin(s)}{4s (1 + s) (t + s)} x(s) ds, \quad a.e \; t \in [0, T] \quad (3.9)$$

has a solution in $L^2[0, T]$. In fact, with the notations of the theorem 4.1, we have

- $p = r = r' = 2, \quad k(t, s) = \frac{t \sin(s)}{4s (1 + s) (t + s)}, \quad b(s) = 0, \quad c = 1.$
- $k(t, \cdot) : s \mapsto \frac{t \sin(s)}{4s (1 + s) (t + s)} \in L^2[0, T].$
- $\Pi : t \mapsto \|k(t, \cdot)\|_2 = \left( \int_0^T \left( \frac{t \sin(s)}{4s (1 + s) (t + s)} \right)^2 ds \right)^{1/2} \in L^\infty [0, T].$

Moreover it is continuous and differentiable over $[0, T]$

- $\|k(t, \cdot)\|_2 \leq \frac{t}{4} \left( \int_0^T \frac{1}{(t + s)^2} ds \right)^{1/2} \leq \frac{T}{4} = M_0 < \frac{1}{4\sqrt{T}} \quad a.e. \; t \in [0, T].$
- The equation $R^2M_0 - R + \sqrt{T} = 0$ has two solutions $R_1 \leq R_2$ which satisfy the assumptions (H5).

By applying theorem 3.1 one concludes that (3.9) has at least one solution in $L^2[0, T]$.

We end this chapter by giving an existence result the more general nonlinear quadratic integral equation (3.2). This is given by the following theorem. we will study the existence of solutions of the quadratic integral equation (3.2) by using the Schauder’s fixed point theorem. This is the subject of the following theorem.

**Theorem 3.3** Let $U$ be the operator defined on $L^p(\mathbb{R}_+)$ by

$$Ux(t) = a(t) + f(t, x(t)) \int_{\mathbb{R}_+} u(t, s, x(s)) ds, \quad (3.10)$$

where $a(\cdot) \in L^p(\mathbb{R}_+)$. Assume that
(h_1) \( f(t, x) : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \) and \( u(t, s, x) : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \) satisfy the Carathéodory’s condition.

(h_2) There exist two positive real numbers \( r \geq 1, c \geq 0 \) and three functions \( k(\cdot, \cdot) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \) and \( m(\cdot), b(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and such that

(i) \( b(\cdot) \in L^r(\mathbb{R}^+) \) and \( s \rightarrow k(t, s) \in L^r(\mathbb{R}_+) \) a.e. \( t \in \mathbb{R}_+ \), where \( \frac{1}{r} + \frac{1}{r'} = 1 \).

(ii) The function \( m(\cdot) \in L^p(\mathbb{R}_+) \), bounded and satisfies \( |f(t, x)| \leq m(t) \) a.e. \( x \in \mathbb{R} \).

(iii) \( |u(t, s, x)| \leq k(t, s)(b(s) + c|x(s)|^{p/r}) \) for any \( t, s \in \mathbb{R}_+ \) and any \( x \in \mathbb{R} \).

(h_3) There exist two positive valued functions \( L_u(\cdot), L_f(\cdot) \) which are continuous in a neighborhood of 0, with \( L_u(0) = L_f(0) = 0 \) and two functions \( g(\cdot) \in L^1(\mathbb{R}_+), h(\cdot) \in L^r(\mathbb{R}_+) \), such that

\[
|f(t, x) - f(\tau, x)| \leq L_f(|t - \tau|), \quad |u(t, s, x) - u(\tau, s, x)| \leq L_u(|t - \tau|) \left( |g(s)| + |h(s)||x|^{p/r} \right),
\]

\( \forall t, s, \tau \in \mathbb{R}_+, x \in \mathbb{R} \).

(h_4) The function \( \psi(t) = m(t)\|k(t, \cdot)\|_{r'} \in L^p(\mathbb{R}_+) \) and we assume that there exists \( R > 0 \) such that \( \|a\|_p + (\|b\|_r + cR^p)\|\psi\|_p \leq R \).

(h_5) There exists a positive and bounded function \( v(\cdot) \) such that \( |f(t, x) - f(t, y)| \leq v(t)|x - y| \) for \( x, y \in \mathbb{R} \) and \( t \geq 0 \). Also, we assume that \( \|v(\cdot)\|_{\infty}\|\psi\|_p(\|b\|_r + cR^{p/r}) < 1 \).

Then, under the above assumptions, equation (3.10) has at least one solution \( x \in L^p(\mathbb{R}_+) \).
For the proof of the above theorem, the reader is referred to [49]. Nonetheless, we note that the main idea in the proof of this theorem is the Shauder’s fixed point theorem applied to a convenient bounded, closed and convex subset of $L^p(\mathbb{R}_+)$. This subset is constructed as follows. Consider the following two positive auxiliary valued functions $\phi(\cdot)$, $E(\cdot)$, defined on $\mathbb{R}_+$ by

$$\phi(t) = |a(t)| + \psi(t) \left( \|b\|_r + cR^{p/r} \right),$$  \hspace{1cm} (3.11)

$$E(\delta) = \frac{2}{1 - \|v(\cdot)\|_\infty \|\psi\|_p (\|b\|_r + cR^{p/r})} \rho(\delta), \quad \delta \geq 0,$$  \hspace{1cm} (3.12)

where

$$\rho(\delta) = \|a(\cdot + \delta) - a(\cdot)\|_p + L_f(\delta)\|\psi(\cdot)\|_p (\|b\|_r + cR^{p/r}) + L_u(\delta) \|m\|_p \left( \|g\|_1 + \|h\|_r' \|\phi\|_p^{p/r} \right), \quad \delta \geq 0.$$  \hspace{1cm} (3.13)

Note that by assumption (h4), it is easy to check that $\phi \in L^p(\mathbb{R}_+)$. Also, since the translation operator $T_\delta$ defined on $L^p(\mathbb{R}_+)$ by $T_\delta x(t) = x(t+\delta)$, $t \geq 0$, is continuous, then we have $\lim_{\delta \to 0} \|a(\cdot + \delta) - a(\cdot)\|_p = 0$. Moreover, by using the conditions on the functions $L_f(\cdot)$ and $L_u(\cdot)$, one can easily check that

$$\lim_{\delta \to 0} E(\delta) = 0.$$  \hspace{1cm} (3.14)

Finally, let $K \subset L^p(\mathbb{R}_+)$, be the subset of $L^p(\mathbb{R}_+)$, defined by

$$K = \{ x \in L^p(\mathbb{R}_+) ; \ |x(t)| \leq \phi(t), \ t \geq 0, \ \text{and} \ |x(\cdot + \delta) - x(\cdot)|_p \leq E(\delta) \}.$$  \hspace{1cm} (3.15)

Then $K$ is the desired bounded, closed and convex subset of $L^p(\mathbb{R}_+)$. 

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Chapter 4

Monotonic solutions of nonlinear integral equations of fractional order

In this chapter, we give existence results for monotonic solutions of the following nonlinear integral equations of fractional order in \( C(\mathbb{R}_+) \).

\[
x(t) = a(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} ds
\]

(4.1)

where \( t \in \mathbb{R}_+, \alpha \in (0, 1) \) and \( \Gamma(\alpha) \) denotes the well-known gamma function. The singular functional integral equations having the form (4.1) are also called equations of fractional order since the term with the integral appearing in (4.1) can be treated from the viewpoint of the concept of Riemann-Liouville integral of fractional order. It is worth while mentioning that integral equations of fractional order create an important subject both of the theory of differential and integral equations and of the so-called fractional calculus (cf. [69], for example). In recent years differential and integral equations of fractional order have found wide applications in physics, mechanics, engineering, electrochemistry, economics and other fields (see [53, 66, 70]). A lot of papers have been devoted to the problem of existence of solutions of nonlinear differential and integral equations of frac-
tional order \[6, 42\]. Let us pay attention to the fact that only a few papers investigated
the existence and properties of solutions of functional integral or differential equations of
fractional order on an unbounded interval \[8, 65\]. In this chapter we are going to prove
a theorem on the existence of solutions of Eq. (4.1) in the space consisting of functions
being defined, continuous and bounded on unbounded interval \(\mathbb{R}^+\). Moreover, we show
that Eq. (4.1) has solutions having limits at infinity. Our goal will be realized with
the help of the technique of measures of noncompactness and a fixed point theorem of
Tychonoff. More precisely, we will apply such a measure of noncompactness so that its
use enables us to prove that Eq. (4.1) has solutions with the mentioned property. Let
us notice that the approach presented in this chapter was never applied in the field of
integral equations of fractional order. In this regard, the results obtained here create a
new subject of the theory of functional integral equations of fractional order.

4.1 Mathematical Preliminaries

In this paragraph, we give some important results concerning the compactness of subsets
of \(C(\mathbb{R}_+)\) as well as measures of noncompactness in \(C(\mathbb{R}_+)\). Finally, we give a continuity
result related to a useful function defined by the fractional order integral operator.

4.1.1 locally convex spaces and measure of noncompactness

**Definition 4.1** Let \(X\) be a topological space. A semi-norm on \(X\) is a map \(p : X \to \mathbb{R}_+\)
such that for all \(x, y \in X\) and \(\lambda \in \mathbb{R}\) or \(\mathbb{C}\):

\[(i)\ x = 0 \Rightarrow p(x) = 0,\]

\[(ii)\ p(\lambda x) = |\lambda|p(x),\]
(iii) \( p(x + y) \leq p(x) + p(y) \).

The only difference with respect to a norm consists in the fact that \( p(x) = 0 \) does not necessarily imply \( x = 0 \).

**locally convex space**: A locally convex space be a vector space \( X \) along with a family of semi-norms \( p \) on \( X \). The space carries a natural topology, the initial topology of the semi-norms. In other words, it is the coarsest topology for which all the semi-norms are continuous.

**Fréchet spaces**: A Fréchet space is a linear space endowed with an invariant metric with respect to translations and is complete with respect to this metric. It is usual to define a Fréchet space by means of the use of semi-norms.

A Fréchet space can be defined as a complete locally convex space with a separated countable family of semi-norms.

By \( C(\mathbb{R}_+) \) we denote the space of continuous functions \( x : \mathbb{R}_+ \rightarrow \mathbb{R} \).

Let \( I_n = [0, n], n \in \mathbb{N} \). The space \( C(\mathbb{R}_+) \) becomes a Fréchet (metric) space equipped with the standard distance.

\[
d(x, y) = \sup \left\{ 2^{-n} \frac{\| x - y \|_n}{1 + \| x - y \|_n} \right\}
\]

for each \( x, y \in C(\mathbb{R}_+) \), where

\[
\| x \|_n := \sup \{ |x(t)| : t \in I_n \}.
\]

**Convergence in \( C(\mathbb{R}_+) \)**: The convergence in \( C(\mathbb{R}_+) \) is the uniform convergence in the compact intervals, i.e. \( x_j \) converge to \( x \) in \( C(\mathbb{R}_+) \) if and only if \( \| x_j - x \|_n \) converge to 0 in \( (C(I_n), \| . \|_n), \forall n \in \mathbb{N} \).

**Compactness in \( C(\mathbb{R}_+) \)**: By Ascoli-Arzela theorem, a set \( M \subset C(\mathbb{R}_+) \) is compact if
and only if for each $n \in \mathbb{N}$, $M$ is a compact set in the Banach space $(C(I_n), \|\cdot\|_n)$ (see [43]).

**Theorem 4.1** (Schauder Tychonoff [29]) Let $\Omega$ be a closed convex subset of a locally convex Hausdorff space $E$. Assume that $H : \Omega \to \Omega$ is continuous and that $\overline{H(\Omega)}$ is compact in $\Omega$. Then $H$ has a fixed point in $\Omega$.

### 4.1.2 Measure of noncompactness in $C(\mathbb{R}_+)$ space

In what follows, we present some basic facts concerning measures of noncompactness in $C(\mathbb{R}_+)$ introduced in [57]. If $X$ is a nonempty subset of $C(\mathbb{R}_+)$ we denote by $\overline{X}$ and $\text{conv}(X)$ the closure and the closed convex closure of $X$, respectively. Next, let us denote by $\mathcal{M}_{C(\mathbb{R}_+)}$ the family of nonempty and bounded subsets of $C(\mathbb{R}_+)$ and by $\mathcal{N}_{C(\mathbb{R}_+)}$ its subfamily consisting of relatively compact subsets.

Let $X \subset \mathcal{M}_{C(\mathbb{R}_+)}$ and fix $T > 0$, $\epsilon > 0$. By $w^T(x, \epsilon)$, we denote the modulus of continuity of the function $x$, that is

$$w^T(x, \epsilon) = \sup \{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \epsilon\}.$$

Further, we put

$$w^T(X, \epsilon) = \sup \{w^T(x, \epsilon) : x \in X\},$$

$$w^T_0(X) = \lim_{\epsilon \to 0} w^T(X, \epsilon),$$

$$w_0(X) = \lim_{T \to \infty} w^T_0(X).$$

It is shown in [57] that $w_0$ satisfies to hypothesis (2)-(6) of the definition of measure of noncompactness (see definition 2.2).
4.1.3 Fractional order integral operators

Definition 4.2 The fractional order integral of order $\alpha \in [0,1]$ of $f \in L^1$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds.$$ 

Where $L^1$ is the space of integrable functions.

Lemma 4.1 Let $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function. Then the function

$$m(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(t,s)}{(t-s)^{1-\alpha}} ds$$

is continuous on the interval $\mathbb{R}_+$.

The proof of this result has been given in [9], for the sake of completeness, we give below its proof.

proof. We fix arbitrarily $t > 0$, then the function $\tau \mapsto g(t,\tau)$ is continuous on the interval $[0,t]$. Hence, the constant $g_t = \sup\{g(t,\tau) : \tau \in [0,t]\}$ is finite. Then we get

$$\int_0^t \frac{g(t,\tau)}{(t-\tau)^{1-\alpha}} d\tau \leq g_t \int_0^t \frac{d\tau}{(t-\tau)^{1-\alpha}} = g_t < \infty.$$

Now, fix arbitrarily $T > 0$, $\epsilon > 0$ and $t,s \in [0,T]$ such that $|t-s| \leq \epsilon$. Without loss of generality we may assume that $t < s$. Then we obtain

$$|m(s) - m(t)| \leq \left| \int_t^s \frac{g(t,\tau)}{(s-\tau)^{1-\alpha}} d\tau - \int_t^s \frac{g(t,\tau)}{(s-\tau)^{1-\alpha}} d\tau \right|$$

$$+ \left| \int_t^s \frac{g(t,\tau)}{(s-\tau)^{1-\alpha}} d\tau - \int_t^s \frac{g(t,\tau)}{(t-\tau)^{1-\alpha}} d\tau \right|$$

$$\leq \int_t^s \frac{g(s,\tau)}{(s-\tau)^{1-\alpha}} d\tau + \left| \int_t^s \frac{g(s,\tau)}{(s-\tau)^{1-\alpha}} d\tau - \int_t^t \frac{g(t,\tau)}{(t-\tau)^{1-\alpha}} d\tau \right|$$

$$+ \left| \int_0^t \frac{g(t,\tau)}{(s-\tau)^{1-\alpha}} d\tau - \int_0^t \frac{g(t,\tau)}{(s-\tau)^{1-\alpha}} d\tau \right|$$

$$\leq g_s \int_t^s \frac{d\tau}{(s-\tau)^{1-\alpha}} + \int_0^t \frac{|g(s,\tau) - g(t,\tau)|}{(s-\tau)^{1-\alpha}} d\tau$$
where we defined

\[ w_T(g, \epsilon) = \left\{ |g(s, \tau) - g(t, \tau)| : t, s, \tau \in [0, T], \tau \leq t, \tau \leq s, |t - s| \leq \epsilon \right\}. \]

Obviously \( \lim_{\epsilon \to 0} w_T(g, \epsilon) = 0 \) which is a consequence of the uniform continuity of the function \( g(t, \tau) \) on the set \( \Delta_T = \{(t, \tau) : t, \tau \in [0, T], \tau \leq t\} \). Linking this assertion with estimate (4.2) we conclude that the function \( m \) is continuous on the interval \([0, T]\). The arbitrariness of \( T \) completes the proof.

### 4.2 Existence results of quadratic integral equations of fractional order

In this section we will discuss the existence of solutions of the two quadratic integral equation of fractional order.

\[ x(t) = a(t) + \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} ds, \quad t \in [0, T], \quad (4.3) \]

and

\[ x(t) = a(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} ds. \quad (4.4) \]
First, we fix two positive real number $T > 0$, $0 \leq \alpha < 1$. In [11], A. Karoui develop an approach for the existence of monotonic and continuous solutions over $[0, T]$ of the following Urysohn’s type nonlinear integral equation of fractional order,

$$x(t) = a(t) + \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} \, ds, \quad t \in [0, T],$$

(4.5)

where $x(.)$ is an unknown function and $a(\cdot) \in C([0, 1])$. Author restrict ourselves to nondecreasing solutions of (4.3), the case of decreasing solutions can be done in a similar way. We should mention that most of the existence results of problem (4.3) have been achieved under the condition that $a(\cdot)$ is nondecreasing on $[0, T]$ as well as other conditions on the function $u(t, s, x)$. In his work, author give an existence proof that requires weaker conditions on $a(\cdot)$ as well as on the function $u(t, s, x)$. In order to solve (4.3), many different methods have been applied in the literature. Most of these methods use the notion of a measure of noncompactness in Banach spaces combined with the Schauder Tychonoff fixed point theorem, see [8, 9, 27, 57]. The existence of solutions of the quadratic integral equation of fractional order (4.3) is based on the Schauder’s fixed point theorem.

**Theorem 4.2** Consider the following nonlinear integral equation

$$x(t) = a(t) + \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} \, ds, \quad t \in [0, T], \quad 0 < \alpha < 1.$$  

(4.6)

Assume that :

$(h_1)$ $a(\cdot) \in C([0, 1])$ and $a(0) \geq 0$.

$(h_2)$ The real valued function $u(t, s, x)$ is continuous on $[0, T]^2 \times \mathbb{R}_+$ and nondecreasing with respect to its three variables, separately.
(h₃) If \(0 \leq t₁ < t₂ \leq T\), then \(a(t₂) - a(t₁) + \frac{u(t₂, t₁, 0)}{\alpha}(t₂ - t₁) ≥ 0\).

(h₄) There exists a function \(ϕ \in C([0, T])\), positive over \([0, T]\) and satisfying the following inequality

\[
a(t) + \int_{0}^{t} \frac{u(t, s, ϕ(s))}{(t - s)^{1-α}} ds ≤ ϕ(t) \quad t \in [0, T].
\]

Then (4.6) has a continuous, positive and nondecreasing solution on \([0, T]\).

The proof of this result has been given in [41], for the sake of completeness, we give below its proof.

**Proof:** We first consider the subset \(Ω_T\) of \(C([0, T])\), given by

\[
Ω_T = \{x(·) \in C([0, T]), 0 \leq x(t) \leq ϕ(t), t \in [0, T]\}.
\]

It is clear that \(Ω_T\) is a nonempty, closed and convex subset of \(C([0, T])\). Moreover, \(Ω_T\) is uniformly bounded by \(∥ϕ∥_∞ = \sup_{t\in[0,T]} |ϕ(t)|\). Consider the integral operator defined on \(C([0, T])\) by \(Fx(t) = a(t) + \int_{0}^{t} \frac{u(t, s, x(s))}{(t - s)^{1-α}} ds\), then we prove that \(F(Ω_T)\) is equicontinuous. Let \(t₁, t₂ \in [0, T]\), we may assume that \(t₁ < t₂\), then we have

\[
|Fx(t₂) - Fx(t₁)| \leq |a(t₂) - a(t₁)| + \left| \int_{0}^{t₁} \frac{u(t₂, s, x(s))}{(t₂ - s)^{1-α}} ds - \int_{0}^{t₁} \frac{u(t₁, s, x(s))}{(t₁ - s)^{1-α}} ds \right|
\]

\[
+ \left| \int_{t₁}^{t₂} \frac{u(t₂, s, x(s))}{(t₂ - s)^{1-α}} ds - \int_{t₁}^{t₂} \frac{u(t₁, s, x(s))}{(t₁ - s)^{1-α}} ds \right| \tag{4.7}
\]

\[
\leq |a(t₂) - a(t₁)| + \left| \int_{0}^{t₁} \frac{u(t₂, s, x(s)) - u(t₁, s, x(s))}{(t₂ - s)^{1-α}} ds \right|
\]

\[
+ \left| \int_{0}^{t₁} \frac{u(t₁, s, x(s))}{(t₂ - s)^{1-α}} ds - \frac{1}{(t₂ - s)^{1-α}} \right| ds
\]

\[
+ \left| \int_{t₁}^{t₂} \frac{u(t₂, s, x(s))}{(t₂ - s)^{1-α}} ds - \frac{1}{(t₁ - s)^{1-α}} \right| ds \tag{4.8}
\]
Since the function \( u(t,s,x) \) is uniformly continuous on \([0,T]^2 \times [0,\|\varphi\|_\infty]\), then we have

\[
\lim_{t_2 \to t_1} |u(t_2,s,x(s)) - u(t_1,s,x(s))| = 0
\]

uniformly in \( s \in [0,T] \) and \( x(\cdot) \in \Omega_T \). Hence, we have

\[
\int_0^{t_2} \frac{|u(t_2,s,x(s)) - u(t_1,s,x(s))|}{(t_2-s)^{1-\alpha}} ds \\
\leq \sup_{s \in [0,T], x \in [0,\|\varphi\|_\infty]} |u(t_2,s,x(s)) - u(t_1,s,x(s))| \frac{t_{2}^{\alpha}}{\alpha} \to 0 \text{ as } t_1 \to t_2. \tag{4.9}
\]

By using the continuity of \( a(t) \) together with (4.7) and (4.9), one concludes that

\[
\lim_{t_1 \to t_2} |F x(t_2) - F x(t_1)| = 0
\]

independently of \( x(\cdot) \in \Omega_T \). Hence, \( F(\Omega_T) \) is an equicontinuous subset of \( C([0,T]) \).

Next, we prove that \( F(\Omega_T) \subset \Omega_T \). Since \( \forall t,s \in [0,T] \), the function \( x \to u(t,s,x) \) is nondecreasing, then \( \forall x(\cdot) \in \Omega_T \), we have

\[
a(t) + \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}} ds \leq a(t) + \int_0^t \frac{u(t,s,\varphi(s))}{(t-s)^{1-\alpha}} ds \leq \varphi(t), \quad t \in [0,T]. \tag{4.10}
\]

The last inequality is due to assumption \((h_4)\). Moreover, by using assumptions \((h_1),(h_2)\) and \((h_3)\), one gets

\[
F x(t) = a(t) + \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}} ds \geq a(t) + \int_0^t \frac{u(t,0,0)}{(t-s)^{1-\alpha}} ds \\
\geq a(t) + u(t,0,0) \frac{t_{\alpha}}{\alpha} = a(0) + \left( a(t) - a(0) + u(t,0,0) \frac{t_{\alpha}}{\alpha} \right) \geq a(0) \geq 0. \tag{4.11}
\]
By using (4.10), (4.11) and the continuity of \(x(\cdot)\), one concludes that \(Fx(\cdot) \in \Omega_T\) whenever \(x(\cdot) \in \Omega_T\). Since \(\Omega_T\) and consequently \(F(\Omega_T)\) is uniformly bounded and since \(F(\Omega_T)\) is equicontinuous, then by Arzela-Ascoli theorem, one concludes that \(F(\Omega_T)\) is a relatively compact subset of \(C([0, T])\).

Next, we prove that \(F : \Omega_T \to \Omega_T\) is continuous. Let \((x_n(\cdot))_n\) be a sequence in \(\Omega_T\) converging to \(x(\cdot)\). Since \(\Omega_T\) is a closed subset of \(C([0, T])\), then \(x(\cdot) \in \Omega_T\). The uniform continuity of the function \(u(\cdot, \cdot, \cdot)\) on \([0, T]^2 \times [0, \|\varphi\|_\infty]\) implies that \(\forall \epsilon > 0, \ \exists \eta > 0\) such that if \(\|x_n(\cdot) - x(\cdot)\|_\infty < \eta\), then we have

\[
\sup_{t \in [0, T]} |Fx_n(t) - Fx(t)| \leq \int_0^t \left( \sup_{t, s \in [0, T]} |u(t, s, x_n(s)) - u(t, s, x(s))| \right) \frac{1}{(t - s)^{1-\alpha}} ds \\
\leq \int_0^T \left( \frac{\alpha}{T^\alpha \epsilon} \right) \frac{1}{(t - s)^{1-\alpha}} ds = \epsilon.
\]

Hence, \(\lim_{n \to +\infty} \|Fx_n - Fx\|_\infty = 0\).

Till now, we have shown that \(F : \Omega_T \to \Omega_T\) is continuous and \(F(\Omega_T)\) is a relatively compact subset of \(C([0, T])\). By using Schauder’s fixed point theorem, one concludes that the operator \(F\) has a fixed point in \(\Omega_T\), that is the problem (4.6) has a positive solution in \(C([0, T])\). It remains to prove that there exists a continuous and monotonic solution of (4.6). To this end, we denote by \(\Gamma_T\) the subset of \(\Omega_T\) given by

\[
\Gamma_T = \{x(\cdot) \in \Omega_T, \ x(\cdot) \text{ is nondecreasing on } [0, T]\}.
\]

Obviously, \(\Gamma_T\) is a closed and convex subset of \(\Omega_T\) and \(F(\Gamma_T)\) is equicontinuous and uniformly bounded. To prove that \(F(\Gamma_T) \subset \Gamma_T\), it suffices to consider \(x(\cdot) \in \Gamma_T\),
\( t_1, t_2 \in [0, T] \) with \( t_1 < t_2 \) and remark that

\[
Fx(t_2) - Fx(t_1) = a(t_2) - a(t_1) + \int_{0}^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} ds \quad - \int_{0}^{t_1} \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} ds
\]

\[
\geq a(t_2) - a(t_1) + \int_{0}^{t_2} u(t_2, s, x(s)) \left( \frac{1}{(t_2 - s)^{1-\alpha}} - \frac{1}{(t_1 - s)^{1-\alpha}} \right) ds
\]

\[
+ \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} ds
\]

Since \( \frac{1}{(t_2 - s)^{1-\alpha}} - \frac{1}{(t_1 - s)^{1-\alpha}} < 0 \), \( x(\cdot) \) is nondecreasing and since \( u(\cdot, \cdot, \cdot) \) is nondecreasing with respect to its variables, then the previous inequality implies

\[
Fx(t_2) - Fx(t_1) \geq a(t_2) - a(t_1) + u(t_2, t_1, x(t_1)) \int_{0}^{t_1} \frac{1}{(t_2 - s)^{1-\alpha}} - \frac{1}{(t_1 - s)^{1-\alpha}} ds
\]

\[
+ u(t_2, t_1, x(t_1)) \int_{t_1}^{t_2} \frac{ds}{(t_2 - s)^{1-\alpha}}
\]

\[
\geq a(t_2) - a(t_1) + u(t_2, t_1, x(t_1)) \left( \frac{t_2^\alpha}{\alpha} - \frac{t_1^\alpha}{\alpha} \right)
\]

\[
\geq a(t_2) - a(t_1) + u(t_2, t_1, 0) \left( \frac{t_2^\alpha}{\alpha} - \frac{t_1^\alpha}{\alpha} \right) \geq 0.
\]

The last inequality is due to assumption \((h_3)\). The previous analysis shows that \( F : \Gamma_T \to \Gamma_T \) satisfies the different conditions of Schauder’s fixed point theorem. Hence, \((4.6)\) has a solution which is continuous, positive and nondecreasing on \([0, T] \).

4.2.1 Examples

Example 1: Let \( T, \beta > 0 \) be arbitrary positive real numbers and consider the following nonlinear integral equation of fractional order and power law nonlinearity

\[
x(t) = a(t) + \int_{0}^{t} \frac{1 + s^\beta(x(s))^\alpha}{(t-s)^{1-\alpha}} ds, \quad 0 < \alpha, < 1, \quad t \in [0, T],
\]

where \( a(t) \in C([0, T]) \) and satisfies the conditions \( a(0) \geq 0 \) and the function \( a(t) + \frac{t^\alpha}{\alpha} \) is nondecreasing. It is clear that under these conditions, \((4.12)\) satisfies conditions \((h_1)\),
(h_2) and (h_3) of the previous theorem. It remains to check that condition (h_4) is also satisfied. To this end, we consider the constant function given by \( \varphi(t) = R_T, \forall t \in [0, T] \), where \( R_T \) is a positive real number satisfying the following condition

\[
\|a\|_{\infty} + \frac{T^\alpha}{\alpha} + R_T^\beta \frac{T^{\alpha+\beta}}{\alpha} \leq R_T.
\]

Note that since \( \lim_{R \to +\infty} \|a\|_{\infty} + \frac{T^\alpha}{\alpha} + R^\beta \frac{T^{\alpha+\beta}}{\alpha} - R = -\infty \), then the above inequality has always a solution. Hence, condition (h_4) is also satisfied. Consequently, (4.12) has a positive, monotonic and continuous solution on \([0, T]\) no matter how large is \( T \).

**Example 2:** Consider the following nonlinear integral equation of fractional order

\[
x(t) = \frac{\sqrt{1+t}}{3} - \frac{\sqrt{t}}{2} + \int_0^t \frac{x^2(s) + 1}{4\sqrt{t-s}} ds, \quad t \in [0, 1].
\]

It is clear that \( a(t) = \frac{\sqrt{1+t}}{3} - \frac{\sqrt{t}}{2} \) satisfies condition (h_1) and it is not a monotone function on \([0, 1]\). Moreover, the function \( u(t, s, x) = \frac{1 + x^2}{4} \) satisfies condition (h_2).

Also, note that if \( 0 \leq t_1 < t_2 \leq 1 \), then

\[
a(t_2) - a(t_1) + u(t_2, t_1, 0)(2\sqrt{t_2} - 2\sqrt{t_1}) = \frac{\sqrt{1+t_2}}{3} - \frac{\sqrt{1+t_1}}{3} \geq 0.
\]

Hence, condition (h_3) is satisfied. To check condition (h_4), it suffices to consider the function \( \varphi(t) = \sqrt{1+t}, t \in [0, 1] \). Straightforward computations show that

\[
a(t) + \int_0^t \frac{\varphi^2(s) + 1}{4\sqrt{t-s}} - \varphi(t) = \frac{\sqrt{t}}{2} + \frac{t^{3/2}}{3} - 2/3 \sqrt{1+t} \leq 0, \quad t \in [0, 1].
\]

Consequently, condition (h_4) is also satisfied. By using the previous theorem, one concludes that (4.13) has a continuous and a nondecreasing positive solution on \([0, 1]\).

In the following paragraph we give an extension existence results given in [57] to the monotonic solutions of quadratic Urysohn’s integral equations. In what follows we will assume that the functions involved in equation (4.1) satisfies the following conditions:
(i) \( a : \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous, bounded and nondecreasing on \( \mathbb{R}_+ \).

(ii) The function \( f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \) is continuous and nondecreasing with respect to each of variables \( t, x \) separately.

(iii) There exists a nondecreasing function \( k : [a_0 = a(0), +\infty[ \to \mathbb{R}_+ \), such that

\[
|f(t, x) - f(t, y)| \leq k(r)|x - y|
\]

for \( t \geq 0 \) and for all \( x, y \in [a_0, r] \)

(iv) \( u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \) is continuous, positive and \( u(t, s, x) \) is nondecreasing with respect to each variable \( t, s \) and \( x \) separately.

(v) There exists a continuous function \( g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) and a continuous and nondecreasing function \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
|u(t, s, x)| \leq g(t, s)\phi(|x|)
\]

for all \( t, s \in \mathbb{R}_+ \) and \( x \in \mathbb{R} \).

(vi) There exists a positive solution \( r_0 \) of the following inequality

\[
\sup_{t \geq 0} |a(t)| + \frac{\phi(r)}{\Gamma(\alpha)} \left[ rk(r) \sup_{t \geq 0} m(t) + \sup_{t \geq 0} m(t)f(t, 0) \right] \leq r
\]

where \( m(t) = \int_0^t \frac{g(t, s)}{(t - s)^{1-\alpha}} ds \), satisfies \( \frac{k(r_0)\phi(r_0)}{\Gamma(\alpha)} \sup_{t \geq 0} m(t) < 1 \).

Now we can formulate our existence result.

**Theorem 4.3** Under assumptions (i)- (vi), equation (4.1) has at least one nondecreasing solution belonging to \( C(\mathbb{R}_+) \).
Proof: For convenience, we write the operator

\[(Hx)(t) = a(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} ds\]

in the form

\[(Hx)(t) = a(t) + (Fx)(t)(Ux)(t), \quad (4.14)\]

where \(F\) is the superposition operator generated by the function \(f(t, x)\) and \(U\) is the operator defined by

\[(Ux)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} ds.\]

Next, let \(r_0\) be a number satisfying assumption (vi) and define the set \(B_{r_0} \subset C(\mathbb{R}_+)\) by

\[B_{r_0} = \{ x \in C(\mathbb{R}_+) : x(t) \geq a_0, \ t \geq 0 \text{ and } \sup_{t \geq 0} |x(t)| \leq r_0 \}.\]

Note that, in view of our assumptions, \(B_{r_0}\) is nonempty, bounded, closed and convex. Moreover, the operators \(F\) and \(U\) are well defined on \(B_{r_0}\). Next, we show that the operators \(F\) and \(U\) transform \(B_{r_0}\) into a subset of \(C(\mathbb{R}_+)\). Indeed, take an arbitrary function \(x \in B_{r_0}, T > 0\) and \(\epsilon > 0\). Let us take arbitrary \(t_1, t_2 \in [0, T]\) such that \(|t_1 - t_2| \leq \epsilon\), then we have

\[|(Fx)(t_2) - (Fx)(t_1)| = |f(t_2, x(t_2)) - f(t_1, x(t_1))| \leq |f(t_2, x(t_2)) - f(t_2, x(t_1))| + |f(t_2, x(t_1)) - f(t_1, x(t_1))| \leq k(r_0)|x(t_2) - x(t_1)| + w^T_{r_0}(f, \epsilon) \leq k(r_0)w^T_{r_0}(x, \epsilon) + w^T_{r_0}(f, \epsilon) \quad (4.15)\]

where \(w^T_{r_0}(f, \epsilon) = \sup \{|f(t_2, x(t_1)) - f(t_1, x(t_1))| : t_1, t_2 \in [0, T], x \in [-r_0, r_0], |t_2 - t_1| \leq \epsilon\}\).

Note that \(w^T_{r_0}(f, \epsilon) \to 0\) as \(\epsilon \to 0\), this is a consequence of the uniform continuity of the
function $f$ on $[0, T] \times [-r_0, r_0]$. We infer that $F$ is continuous on $[0, T]$ for any $T > 0$. This implies that $Fx$ is continuous on $\mathbb{R}_+$. Therefore $F$ transforms $B_{r_0}$ into a subset of $C(\mathbb{R}_+)$. We show that the same holds for the operator $U$. Let $T > 0$ and $t_1, t_2 \in [0, T]$. Without loss of generality, we may assume that $t_1 \leq t_2$. Then we have

$$|(Ux)(t_2) - (Ux)(t_1)| = \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} ds \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_2 - s)^{1-\alpha}} ds \right| + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_2} \frac{u(t_1, s, x(s))}{(t_2 - s)^{1-\alpha}} ds \right|$$

Keeping in mind assumption (v), we obtain

$$|(Ux)(t_2) - (Ux)(t_1)| \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{w_T^{r_0}(u, \epsilon)}{(t_2 - s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} g(t_1, s) \phi(r_0) \left[ (t_1 - s)^{1-\alpha} - (t_2 - s)^{1-\alpha} \right] ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{w_T^{r_0}(u, \epsilon)}{(t_2 - s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \phi(r_0) g_T \int_0^{t_1} \left[ (t_1 - s)^{1-\alpha} - (t_2 - s)^{1-\alpha} \right] ds$$

$$+ \frac{1}{\Gamma(\alpha)} \phi(r_0) g_T \int_0^{t_1} \frac{ds}{(t_2 - s)^{1-\alpha}} + \frac{1}{\Gamma(\alpha + 1)} \left[ w_T^{r_0}(u, \epsilon) T^\alpha + 2 \phi(r_0) g_T \epsilon^\alpha \right].$$

(4.16)
Here,

\[ w^T_{r_0}(u, \epsilon) = \sup \{ |u(t_2, s, x(s)) - u(t_1, s, x(s))| : t_1, t_2, s \in [0, T], |t_1 - t_2| \leq \epsilon, x \in [-r_0, r_0] \} \]

and

\[ g_T = \sup \{ g(t, s) : (t, s) \in [0, T] \times [0, T] \} . \]

Observe that by invoking the uniform continuity of the function \( u(t, s, x) \) on \([0, T]^2 \times [-r_0, r_0]\), we obtain

\[ \lim_{\epsilon \to 0} w^T_{r_0}(u, \epsilon) = 0. \]

Keeping in mind estimate (4.16), we conclude that the function \( Ux \) is continuous on \([0, T] \times \mathbb{R}_+\). Finally, combining the continuity of the functions \( Fx \) and \( Ux \), we deduce that the function \( Hx \) is continuous on \( \mathbb{R}_+ \). Moreover for an arbitrary \( x \in B_{r_0} \) and \( t \geq 0 \), we obtain

\[
(Hx)(t) \leq |a(t)| + \frac{1}{\Gamma(\alpha)} \left[ |f(t, x(t)) - f(t, 0)| + f(t, 0) \right] \int_0^t \frac{|u(t, s, x(s))|}{(t - s)^{1-\alpha}} ds \\
\leq |a(t)| + \frac{1}{\Gamma(\alpha)} [r_0 k r_0 + f(t, 0)] \phi(r_0) \int_0^t \frac{g(t, s)}{(t - s)^{1-\alpha}} ds \\
\leq |a(t)| + \frac{\phi(r_0)}{\Gamma(\alpha)} [r_0 k r_0 + f(t, 0)] m(t).
\]

In view of assumption (vi), we deduce that \( \sup_{t \geq 0} |Hx(t)| \leq r_0 \). Moreover, by assumptions (i), (ii) and (iv), it is easy to see that \( (Hx)(t) \geq a(0), \forall t \geq 0 \). The previous analysis shows that \( H \) transforms \( B_{r_0} \) into itself. Now, let us consider the subset \( \Omega \) of \( B_{r_0} \) consisting of all functions from \( B_{r_0} \) which are nondecreasing on \( \mathbb{R}_+ \). Observe that \( \Omega \) is nonempty, bounded, closed and convex. Moreover, \( H \) transforms \( \Omega \) into a subset of \( C(\mathbb{R}_+) \). Let us show that \( H \) transforms \( \Omega \) into itself. Let \( x \in \Omega \) and fix \( t_1, t_2 \in \mathbb{R}_+ \).
with \( t_1 \leq t_2 \), then we have

\[
(Hx)(t_2) - (Hx)(t_1) = a(t_2) - a(t_1) \\
+ \frac{f(t_2, x(t_2))}{\Gamma(\alpha)} \int_0^{t_2} u(t_2, s, x(s)) \, ds - \frac{f(t_1, x(t_1))}{\Gamma(\alpha)} \int_0^{t_1} u(t_1, s, x(s)) \, ds
\]

\[
= a(t_2) - a(t_1) \\
+ \frac{f(t_2, x(t_2))}{\Gamma(\alpha)} \int_0^{t_2} u(t_2, s, x(s)) \, ds - \frac{f(t_1, x(t_1))}{\Gamma(\alpha)} \int_0^{t_1} u(t_2, s, x(s)) \, ds
\]

\[
+ \frac{f(t_1, x(t_1))}{\Gamma(\alpha)} \int_0^{t_1} u(t_1, s, x(s)) \, ds - \frac{f(t_1, x(t_1))}{\Gamma(\alpha)} \int_0^{t_1} u(t_1, s, x(s)) \, ds
\]

In view of assumptions (ii) and (iv), we have

\[
\left[ \frac{f(t_2, x(t_2)) - f(t_1, x(t_1))}{\Gamma(\alpha)} \right] \int_0^{t_2} u(t_2, s, x(s)) \, ds \geq 0. \tag{4.17}
\]

It remains to show that

\[
\int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2-s)^{1-\alpha}} \, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1-s)^{1-\alpha}} \, ds \geq 0. \tag{4.18}
\]

This is done as follows. Since

\[
\int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2-s)^{1-\alpha}} \, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1-s)^{1-\alpha}} \, ds = \int_0^{t_1} \frac{u(t_2, s, x(s))}{(t_2-s)^{1-\alpha}} \, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1-s)^{1-\alpha}} \, ds
\]

\[
+ \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2-s)^{1-\alpha}} \, ds,
\]

then taking into account the assumption (iv), we obtain

\[
\int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2-s)^{1-\alpha}} \, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1-s)^{1-\alpha}} \, ds
\]

\[
\geq \int_0^{t_1} u(t_2, s, x(s)) [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] \, ds
\]
\[ + \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} ds \]

\[ \geq u(t_2, t_1, x(t_1)) \int_{0}^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] ds \]

\[ + u(t_2, t_1, x(t_1)) \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \]

\[ \geq u(t_2, t_1, x(t_1)) \left[ \int_{0}^{t_2} (t_2 - s)^{\alpha-1} ds - \int_{0}^{t_1} (t_1 - s)^{\alpha-1} ds \right] \]

\[ \geq \frac{u(t_2, t_1, x(t_1))}{\alpha} (t_2^{\alpha} - t_1^{\alpha}) \geq 0 \quad (4.19) \]

This implies that \((Hx)(t_2) - (Hx)(t_1)\geq 0\). Therefore \(Hx\) is nondecreasing on \(\mathbb{R}_+\) and \(H(\Omega) \subset \Omega\). Now, let us take a nonempty set \(X \subset \Omega\). Fix \(T > 0, \epsilon > 0\) and choose \(x \in X\) and \(t_1, t_2 \in [0, T]\) such that \(|t_1 - t_2| < \epsilon\). Then, by using (5.6) and (4.16), we obtain

\[ |(Hx)(t_2) - (Hx)(t_1)| \leq |a(t_2) - a(t_1)| + (Fx)(t_2)\{(Ux)(t_2) - (Ux)(t_1)| \]

\[ + (Ux)(t_2)|((Fx)(t_2) - (Fx)(t_1)| \]

\[ \leq w^T(a, \epsilon) + [r_0 k(r_0) + f(t_2, 0)] \frac{1}{\Gamma(\alpha + 1)} \left[ w^T_{r_0}(u, \epsilon) T^\alpha + 2 \phi(r_0) gT \epsilon^\alpha \right] \]

\[ + \frac{\phi(r_0)}{\Gamma(\alpha)} m(t_2) [k(r_0) w^T(x, \epsilon) + w^T(f, \epsilon)]. \quad (4.21) \]

Now, using the uniform continuity of the functions \(f\) and \(u\) on \([0, T] \times [-r_0, r_0]\) and \([0, T] \times [0, T] \times [-r_0, r_0]\) respectively, we get

\[ w^T_0(Hx, \epsilon) \leq \frac{k(r_0) \phi(r_0)}{\Gamma(\alpha)} \sup_{t \leq T} m(t) w^T_0(x, \epsilon) \]

and

\[ w_0(HX) \leq \frac{k(r_0) \phi(r_0)}{\Gamma(\alpha)} \sup_{t \geq 0} m(t) w_0(X, \epsilon) \quad (4.22) \]

Next, we show that \(H\) is continuous on the set \(\Omega\). Let \((x_n)_n \subset \Omega\) be a sequence converging to \(x\) and fix \(T > 0\). We show that \(|Hx_n - Hx|\) converges uniformly to 0 on \([0, T]\). Since

\[ |Hx_n(t) - Hx(t)| \quad (4.23) \]
\begin{align*}
&\leq \frac{|f(t, x_n(t)) - f(t, x(t))|}{\Gamma(\alpha)} \int_0^T \frac{|u(t, s, x_n(s))|}{(t-s)^{1-\alpha}} ds \\
&+ \frac{|f(t, x_n(t))|}{\Gamma(\alpha)} \int_0^T \frac{|u(t, s, x_n(s)) - u(t, s, x(s))|}{(t-s)^{1-\alpha}} ds \\
&\leq k(r_0)|x_n(t) - x(t)| \frac{\phi(r_0)}{\Gamma(\alpha)} m(t) \\
&+ \frac{(r_0k(r_0) + f(t, 0))}{\Gamma(\alpha)} \int_0^T \frac{|u(t, s, x_n(s)) - u(t, s, x(s))|}{(t-s)^{1-\alpha}} ds \quad (4.24)
\end{align*}

and since sup_{t\in[0,T]} m(t) < +\infty, then

$$
\lim_{n \to \infty} \left[ k(r_0)|x_n(t) - x(t)| \frac{\phi(r_0)}{\Gamma(\alpha)} m(t) \right] = 0
$$

uniformly on [0, T]. On the other hand, from the continuity of u

$$
|u(t, s, x_n(s)) - u(t, s, x(s))| \leq w_0^T(u, \sup_{s\in[0,T]} |x_n(s) - x(s)|) \quad (4.25)
$$

where

$$
w_0^T(u, \epsilon) = \sup \{|u(t, s, x) - u(t, s, y)| : t, s \in [0, T], |x - y| \leq \epsilon\}.
$$

From the fact that

$$
\sup_{t\in[0,T]} (r_0k(r_0) + f(t, 0)) < \infty,
$$

one concludes that

$$
\lim_{n \to \infty} |Hx_n(t) - Hx(t)| = 0
$$

uniformly on [0, T]. Since T is arbitrary, then one that H : \Omega \to \Omega, is continuous. To use the Tychonoff fixed point theorem, we construct an appropriate subset \( \tilde{Q} \) of \( \Omega \) as follows. Let \((\Omega_n)_n\) be a sequence of subsets of \( \Omega \) defined by

\[
\begin{cases}
\Omega_1 = \overline{\text{conv}(H(\Omega))} \\
\Omega_n = \overline{\text{conv}(H(\Omega_{n-1}))} \quad \text{for} \ n \geq 1.
\end{cases}
\]

Observe that, \( \Omega_n \subset \Omega_{n-1} \) for all \( n \geq 1 \). Moreover, straightforward computations show that \( w_0(\Omega) \leq w_0(B_{r_0}). \) Moreover, we have \( w_0(B_{r_0}) = 2r_0 \), see [57]. Let \( \tilde{Q} = \bigcap_{n \geq 1} \Omega_n \). It is
clear that $\tilde{Q}$ is nonempty, closed and convex and $H(\tilde{Q}) \subset \tilde{Q}$. It follows from (vi) and the definition of $w_0$ that

$$w_0(\tilde{Q}) = w_0(\bigcap_{n \geq 1} \Omega_n) = \lim_{n \to \infty} w_0(\Omega_n) = \lim_{n \to \infty} w_0(H(\Omega_{n+1})) \leq \lim_{n \to \infty} k^n w_0(\Omega) = 0,$$

where $k = \frac{k(r_0)\phi(r_0)}{\Gamma(\alpha)} \sup_{t \geq 0} m(t) < 1$. This proves that $\tilde{Q} \subset \mathfrak{N}_{C(\mathbb{R}_+)}$. That is $H(\tilde{Q})$ is compact in $C(\mathbb{R}_+)$. Finally, the Tychonoff fixed point theorem implies that $H$ has a fixed point in $\tilde{Q}$. This completes the proof of the previous theorem.
Chapter 5

Measure of noncompactness and Mild solutions of functional evolution equations.

The purpose of this last chapter is to apply the tool of measure of noncompactness fixed points based theorem to give an alternative approach for the existence of solution of integrodifferential evolution equation in Banach space $E$ of the form

$$\begin{cases} 
  x'(t) = A(t)x(t) + f\left(t, x(t), \int_0^t u(t, s, x(s)) ds\right), & t \geq 0, \\
  x(0) = x_0.
\end{cases} \tag{5.1}$$

Where $A(t) : D_t \subset E \to E$ is an infinitesimal generator of an analytic semigroup of bounded linear operators $U(t, s)$ and $f : \mathbb{R}_+ \times E \to E$ is a given function.

In the past few years, several books have been devoted to the study the existence on compact intervals of mild solutions for differential equations in abstract spaces. See for example [36, 38, 62]. In this chapter, we prove a theorem on the existence of mild solutions for the semilinear integrodifferential equation (5.1) on an unbounded interval.
The proof of this result is largely inspired from a recent work of Olszowy et al \[60\]. The considerations of this application are based on the notion of measure of noncompactness in the space of all functions continuous on $\mathbb{R}_+$ (see \[58, 60\]). In order to prove this existence result, we shall rely on Tichonov fixed point theorem.

5.1 Preliminary tools

In what follows, $E$ will represent a Banach space with norm $\| \cdot \|$. $C(\mathbb{R}_+, E)$ the space of continuous functions $x : \mathbb{R}_+ \to E$. Let $I_n = [0, n], n \in \mathbb{N}$. The space $C(\mathbb{R}_+)$ is the locally convex space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$ with the metric

$$d(x, y) = \sup \left\{ 2^{-n} \frac{\|x - y\|_n}{1 + \|x - y\|_n} \right\},$$

where

$$\|x\|_n := \sup \{ |x(t)| : t \in I_n \}.$$

The convergence in $C(\mathbb{R}_+)$ is the uniform convergence in the compact intervals, i.e. $x_j$ converge to $x$ in $C(\mathbb{R}_+, E)$ if and only if $\|x_j - x\|_n$ converge to 0 in $(C(I_n), \| \cdot \|_n), \forall n \in \mathbb{N}$. By Arzela-Ascoli theorem, a set $M \subset C(\mathbb{R}_+, E)$ is compact if and only if for each $n \in \mathbb{N}$, $M$ is a compact set in the Banach space $(C(I_n), \| \cdot \|_n)$, see \[43\].

Next, we present some basic facts concerning measure of noncompactness in $C(\mathbb{R}_+, E)$ (see \[12, 57, 58\]). let be $\theta$ the zero element of $E$. Denote by $B(x, r)$ the closed ball centred at $x$ and with radius $r$ and by $B_r$ the ball $B(x, r)$. If $X$ is a nonempty subset of $E$ we denote by $\overline{X}$, $\text{conv}(X)$ the closure and convex closure of $X$, respectively. Finally, let us denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of $E$ and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets. Following \[12\] we accept the following definition of the concept of a measure of noncompactness:
Definition 5.1 A function \( \mu : \mathcal{M}_E \to \mathbb{R}_+ \) is said to be a measure of noncompactness in \( E \) if it satisfies the following conditions:

1. The family \( \ker \mu = \{ X \in \mathcal{M}_E : \mu(X) = 0 \} \) is nonempty and \( \ker \mu \subset \mathcal{N}_E \).

2. \( X \subset Y \Rightarrow \mu(X) \leq \mu(Y) \).

3. \( \mu(\overline{X}) = \mu(\text{Conv}X) = \mu(X) \).

4. \( \mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y) \) for \( \lambda \in [0, 1] \).

5. If \( \{X_n\}_n \) is a sequence of nonempty, bounded, closed subsets of \( C(\mathbb{R}_+) \) such that \( X_{n+1} \subset X_n \) for \( n = 1, 2, \ldots \) and \( \lim_{n \to \infty} \mu(X_n) = 0 \) then the set \( X_\infty = \bigcap_{n=1}^{\infty} X_n \) is nonempty.

Next, we consider the measure of noncompactness in \( C(\mathbb{R}_+, E) \) defined in \([57, 58, 60]\) as follows. Let

\[
\mathcal{M}_r = \{ X \subset C(\mathbb{R}_+, E) : X \neq \emptyset \text{ and } \|x(t)\| \leq r(t) \text{ for } x \in X \text{ and } t \geq 0 \}
\]

and let \( \mathcal{N}_r \) be the family of all relatively compact subsets of \( \mathcal{M}_r \). Fix \( X \in \mathcal{M}_E \) and a positive number \( T > 0 \). For \( x \in X \) and \( \epsilon > 0 \), denote by \( w^T(x, \epsilon) \) the modulus of continuity of the function \( x \) on the interval \( [0, T] \), i.e.

\[
w^T(x, \epsilon) = \sup \{ \|x(t) - x(s)\| : t, s \in [0, T], |t - s| \leq \epsilon \}.
\]

Further, let us put

\[
w^T(X, \epsilon) = \sup \{ w^T(x, \epsilon) : x \in X \},
\]

\[
w_0^T(X) = \lim_{\epsilon \to 0} w^T(X, \epsilon).
\]
now, let $\mu$ be a regular measure of noncompactness in $E$ and let

$$
\mu^T(X) = \sup\{\mu(X(t)) : t \in [0, T]\}.
$$

(5.2)

Assume that there exists a function $R : \mathbb{R}_+ \to (0, \infty)$ such that $R(t) \geq r(t)$ for $t \geq 0$.

Define the mapping $\gamma_R$ on the family $\mathcal{M}_r$ by

$$
\gamma_R(X) = \sup \left\{ \left( \frac{1}{R(t)} \right) \left( w_t^T(X) + \mu^T(X) \right), T \geq 0 \right\}
$$

(5.3)

The properties of $\gamma_R$ is given by the following theorem

**Theorem 5.1** [60] The mapping $\gamma_R : \mathcal{M}_r \to \mathbb{R}_+$ satisfies the conditions

1. The family $\ker \gamma_R = \{ X \in \mathcal{M}_r : \gamma_R(X) = 0 \} = \mathcal{N}_r$.

2. $\gamma_R(\text{conv}(X)) = \gamma_R(X)$.

3. If $(X_n)$ is a sequence of closed sets from $\mathcal{M}_{r_0}$ such that $X_{n+1} \subset X_n$, ($n = 0, 1, \ldots$) and if $\lim_{n \to \infty} \gamma_R(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Let $d$ be the metric associated with the norm $\|\cdot\|$ in $E$ and $X \subset E$ be a bounded subset.

By $d(x, X)$, we denote the distance between point $x$ and the set $X$.

**Definition 5.2** Let $X, Y \subset E$ be two nonempty and bounded sets. The number

$$
d_H(X, Y) = \max \left\{ \sup_{x \in X} d(x, X), \sup_{y \in Y} d(y, Y) \right\}
$$

is called the Hausdorff distance between $A$ and $B$.

**Lemma 5.1** [12] If $\mu$ is a regular measure of noncompactness, then

$$
|\mu(X) - \mu(Y)| \leq \mu(B(\theta, 1))d_H(X, Y)
$$

for any bounded subsets $X, Y$ of $E$, $d_H$ is the Hausdorff distance between $X$ and $Y$. 

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The following lemmas borrowed from [12, 60] will be needed in the proof of our existence result of solution of (5.1).

**Definition 5.3** [33] A two parameters family of bounded linear operators $U(t,s)$ ($0 \leq s \leq t$), on $E$ is called an evolution system if the following conditions are satisfied

1. $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$, for $0 \leq s \leq r \leq t$,

2. $(t,s) \mapsto U(t,s)$ is strongly continuous for $0 \leq s \leq t$.

**Lemma 5.2** If $\mu$ is a regular measure of noncompactness, $U(t,s)$, $0 \leq s \leq t \leq T$ is an evolution system, and $X \subset E$ is nonempty and bounded set, then

$$\mu(U(t,s)X) \leq \mu(X).$$

**Lemma 5.3** [60] If all functions belonging to $X$ are equicontinuous on compact subsets of $\mathbb{R}_+$ then

$$\mu\left(\int_a^t X(s)ds\right) \leq \int_a^t \mu(X(s))ds.$$

**Lemma 5.4** (Cauchy’s formula) If $f : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function then

$$\int_a^t \cdots \int_a^{s_n} f(s_{n+1})ds_{n+1}ds_n\cdots ds_1 = \frac{1}{n!} \int_a^t f(s)(t-s)^n ds \text{ for each } t \geq a$$

Our consideration are based on following Tikhonov fixed point theorem

**Theorem 5.2** [25] Let $K$ be a closed convex subset of locally convex Hausdorff space $E$. Assume that $F : K \to K$ is continuous and that $F(K)$ is relatively compact in $E$. Then $F$ has at least one fixed point in $K$. 

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5.2 Existence of mild solutions

In this section by using the usual technique of measure of noncompactness and its application in differential equations in Banach space (see [57]), we give an existence result for the problem (5.1). The following hypotheses will be needed in the sequel.

(A) $A(t)$ is a bounded linear operator on $E$ for each $t \geq 0$ and generates a uniformly continuous evolution system $U(t, s)$.

(C_f) (i) $(t, x) \mapsto f(t, x, y)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times E$ with value in $E$ and there exists $p \in L^2(\mathbb{R}_+, \mathbb{R}_+)$ such that $\|f(t, x, y)\| \leq p(t)\left(\|x\| + \|y\|\right)$ for a.e. $t \geq 0$ and all $x \in E$.

(ii) There exists $h_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ measurable and essentially bounded function on compact intervals of $\mathbb{R}_+$ and satisfies that

$$
\mu(f(t, X, Y)) \leq h_1(t) \max\left(\mu(X), \mu(Y)\right)
$$

for a.e $t \in \mathbb{R}_+$ and for bounded subsets $X, Y$ of $E$.

(C_u) (i) $u(t, s, x) : \mathbb{R}_+ \times \mathbb{R}_+ \times E \rightarrow E$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times E$.

(ii) $k : (t, s) \in \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfies $t \mapsto \int_0^t k(t, s)ds \in L^2(\mathbb{R}_+, \mathbb{R}_+) \ (0 \leq s \leq t)$ such that $\|u(t, s, x)\| \leq k(t, s)\phi(\|x\|)$, where $\phi : \mathbb{R}_+ \rightarrow (0, \infty)$ is continuous and increasing with $\phi(0) = 0$.

(iii) There exists $h_2 : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ measurable and essentially bounded function on compact intervals of $\mathbb{R}_+ \times \mathbb{R}$ such that

$$
\mu(u(t, s, X)) \leq h_2(t, s)\mu(X)
$$

for a.e. $(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+$ and all bounded subsets $X$ of $E$. 78
Definition 5.4 A continuous function \( x : \mathbb{R}_+ \to E \) is said to be a mild solution of (5.1) if \( x \) satisfies to
\[
x(t) = U(t)x_0 + \int_0^t U(t,s)f\left(s,x(s),\int_0^s u(s,\tau,x(\tau))d\tau\right)ds.
\] (5.4)

Our main result is given by the following theorem which is the analogue of theorem 3.2 in [60] to the functional integrodifferential case. Its proof follows the same lines of theorem 3.2 in [60], with the necessary modifications and extra work to treat the more general functional case.

Theorem 5.3 Let \( E \) be a Banach space. Assume that the hypotheses \( A, C_f \) and \( C_g \) are satisfied. Then for each \( x_0 \in E \), the problem (5.1) has at least one mild solution \( x \) in \( C(\mathbb{R}_+,E) \), for \( t \geq 0 \).

Proof. Define the operator \( F : C(\mathbb{R}_+,E) \to C(\mathbb{R}_+,E) \) by
\[
(Fx)(t) = U(t)x_0 + \int_0^t U(t,s)f\left(s,x(s),\int_0^s u(s,\tau,x(\tau))d\tau\right)ds, \quad t \geq 0.
\] (5.5)

For \( x \in C(\mathbb{R}_+,E) \). Let
\[
r(t) = G^{-1}\left(M \int_0^t p(s)\tilde{k}(s)ds\right), \quad M = \sup\{\|T(t)\| : t \geq 0\}, \quad \tilde{k}(s) = \max(1,\int_0^s k(s,\tau)d\tau)
\]
and
\[
G(z) = \int_{M\|x_0\|}^z \frac{dz}{z + \phi(z)}.
\]

First, we shall show that if \( x \in C(\mathbb{R}_+,E) \) and \( \|x(t)\| \leq r(t) \) for \( t \geq 0 \) then
\[
\|(Fx)(t)\| \leq r(t), \quad \text{for } t \geq 0.
\]

Applying assumptions \( C_f(i) \) and \( C_g(ii) \) we have
\[
\|(Fx)(t)\| \leq \|U(t)x_0\| + \left\| \int_0^t U(t,s)f\left(s,x(s),\int_0^s u(s,\tau,x(\tau))\right)ds \right\|
\]
From the estimate (5.6), we deduce that $F$ transforms $\Omega$ into itself. In what follows we will estimate the modulus of continuity of the function $Fx$. To do this let us fix $x \in C(\mathbb{R}_+, X)$ such that $\|x(t)\| \leq r(t)$. Fix an arbitrary $T \geq 0$ and $\epsilon \geq 0$ and let $t_1, t_2 \in [0, T]$ such that $|t_1 - t_2| \leq \epsilon$. Without loss of generality, we may assume that $t_1 \leq t_2$. Then, in view of our assumptions we get:

$$
\| (Fx)(t_2) - (Fx)(t_1) \| \leq \left\| \left( U(t_2, 0) - U(t_1, 0) \right) x_0 \right\|
+ \int_{t_1}^{t_2} \| U(t_2, s) \| \left[ p(s) \left( r(s) + \int_0^s k(s, \tau) \phi(r(\tau)) d\tau \right) \right] ds
+ \int_0^{t_1} \| U(t_2, 0) - U(t_1, 0) \| \left[ p(s) \left( r(s) + \int_0^s k(s, \tau) \phi(r(\tau)) d\tau \right) \right] ds
\leq w^T(U(., 0), \epsilon) \| x_0 \| + \nu^T(U, \epsilon) \int_0^{t_1} \int_0^s p(s) \left( r(s) + k(s, \tau) \phi(r(\tau)) d\tau \right) ds
+ M \int_{t_1}^{t_2} \int_0^s p(s) \left( r(s) + k(s, \tau) \phi(r(\tau)) d\tau \right) ds. \tag{5.7}
$$

Putting

$$
\Delta(U) = w^T(U(., 0), \epsilon) \| x_0 \| + \nu^T(U, \epsilon) \int_0^T p(s) \left( r(s) + \int_0^T k(s, \tau) \phi(r(\tau)) d\tau \right) ds
+ M \sup \left\{ \int_{t_1}^{t_2} p(s) \left[ r(s) + \int_0^s k(s, \tau) \phi(r(\tau)) d\tau \right] ds : s \leq t \leq T, \ |t - s| \leq \epsilon \right\},
$$

where

$$
w^T(U(., 0), \epsilon) = \sup \{ \| U(t_2, 0) - U(t_1, 0) \| : t_1, t_2 \in [0, T], |t_1 - t_2| \leq \epsilon \},
$$
\[ \nu^T(U(-s), \epsilon) = \sup \{ \|U(t_2, s) - U(t_1, s)\| : 0 \leq t_1 \leq t_2 \leq T, |t_2 - t_1| \leq \epsilon \}. \]

Then, we have

\[ \| (Fx)(t_2) - (Fx)(t_1) \| \leq \Delta(U, \epsilon), \text{ for } x \text{ such that } \|x(t)\| \leq r(t). \] (5.8)

Under assumptions (A), \( C_f(i) \) and \( C_g(i) - (ii) \), we have

\[ \lim_{\epsilon \to 0} \Delta(U, \epsilon) = 0. \]

Next, define the subset

\[ \Omega_T = \left\{ x \in C(\mathbb{R}_+, X) : \|x(t)\| \leq r(t) \text{ and } w^T(x, \epsilon) \leq \Delta(x, \epsilon) \right\}. \]

Since \( x_0 \in \Omega_T \), then \( \Omega_T \) is nonempty. Moreover, it is easy to see that \( \Omega \) is a bounded, closed and convex subset of \( C(\mathbb{R}_+, E) \). By the estimate (5.6), \( F \) transforms \( \Omega_T \) into itself.

In what follows we show that \( F : \Omega_T \to \Omega_T \) is continuous. Let \( x, x_n \in \Omega_T \) such that \( x_n \to x \in \Omega_T \). The property of convergence in \( C(\mathbb{R}_+, X) \) implies that

\[ \lim_{n \to \infty} \sup_{t \leq T} \|x_n(t) - x(t)\| = 0, \text{ for } T \geq 0. \]

Fix \( T \geq 0 \). Observe that

\[ \|f\left(t, x_n(t), \int_0^t u(t, s, x_n(s))ds\right) - f\left(t, x(t), \int_0^t u(t, s, x(s))ds\right)\| \leq 2\phi(r(T))p(t)\hat{k}(t) \in L^1(\mathbb{R}_+). \]

Then, by Lebesgue’s dominated convergence theorem we have

\[ \lim_{n \to \infty} \sup_{t \leq T} \| (Fx_n)(t) - (Fx)(t) \| \leq M \lim_{n \to \infty} \sup_{t \leq T} \int_0^t \int_0^s \left( \|f(s, \tau, x_n(\tau))\| - \|f(s, \tau, x(\tau))\| \right) d\tau ds \]

\[ = 0. \] (5.9)

In order to complete our proof, let us defined the sequence \((Q_n)\) of subsets of \( C(\mathbb{R}_+, E) \) by

\[ \begin{cases} Q_0 = \Omega_T, \\ Q_n = \text{conv}(F(Q_{n-1})) \text{ for } n \in \mathbb{N} \setminus \{0\}. \end{cases} \] (5.10)
Observe that all subsets of this sequence are nonempty, closed and convex. Moreover,

\[ Q_{n+1} \subset Q_n, \quad \text{for} \quad n \in \mathbb{N}. \]

The equicontinuity of the set \( \Omega \) on compact intervals, implies that

\[ w^T_n(Q_n) = 0 \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad T \geq 0. \] (5.11)

Next, defined the sequence \( (z_n) \in C(\mathbb{R}_+, E) \) by \( z_n(t) = \mu(Q_n)(t) \). Obviously \( 0 \leq z_{n+1}(t) \leq z_n(t), \quad (n = 0, 1, \ldots) \). thus the sequence converge uniformly to the function \( z_\infty(t) \). By, lemma 5.1 and (5.8) we get

\[ |z_n(t) - z_n(s)| \leq \mu(B(\theta,1))\Delta(T,|t-s|) \]

which implies the continuity of \( z_n \) on \( \mathbb{R}_+ \). Using Lemma 5.8 (C)(i), and the properties of the measure of noncompactness \( \mu \), we obtain

\[
\begin{align*}
z_n(t) &= \mu\left(\text{conv}(FQ_n)(t)\right) \\
&\leq \mu\left(\int_0^t U(t,s)f\left(s,Q_{n-1}(s),\int_0^s u(s,\tau,z_n(Q_{n-1}(\tau))d\tau\right)ds\right) \\
&\leq M \int_0^t h_1(s) \max\left(\mu(Q_{n-1}(s)),\mu(\int_0^s u(s,\tau,z_n(Q_{n-1}(\tau))d\tau\right)ds. \quad (5.12)
\end{align*}
\]

Let us consider the two following cases.

**Case 1:** \( \mu(Q_{n-1}(s)) \leq \mu\left(\int_0^s u(s,\tau,z_n(Q_{n-1}(\tau))d\tau\right) \). Then

\[
\begin{align*}
z_n(t) &\leq M \int_0^t h_1(s)h_2(s,\tau)z_n(Q_{n-1}(\tau))d\tau ds \\
&\leq M \int_0^t \int_0^s h_1(s)h_2(s,\tau)z_n(Q_{n-1}(\tau))d\tau ds. \quad (5.13)
\end{align*}
\]

Next, by applying lemma 5.4 we have

\[
z_{n+1}(t) \leq M^{n+1} \int_0^t \int_0^s h_1(s)h_2(s,\tau_1) \int_0^{\tau_1} \int_0^{\tau_2} h_1(s_1)h_2(s_1,\tau_2) ... \]

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\[
\ldots \int_{0}^{s_n} \int_{0}^{s_n} h_1(s_1)h_2(s_n, \tau_{n+1})z_0(\tau_n)d\tau_1 \ldots d\tau_n ds_1 \ldots ds_n \\
\leq M^n [\overline{h}_1(t)\overline{h}_2(t)]^{n+1} \int_{0}^{t} \int_{0}^{s_1} \ldots \int_{0}^{s_n} \left[ \int_{0}^{s} \int_{0}^{\tau_1} \ldots \int_{0}^{\tau_n} z_0(\tau_{n+1})d\tau_1 \ldots d\tau_{n+1} \right] ds_1 \ldots ds_n \\
\leq M^n [\overline{h}_1(t)\overline{h}_2(t)]^{n+1} t^{n+1} \frac{(s - \tau)^n}{n!} z_0(\tau)d\tau \\
\leq \frac{M^n [\overline{h}_1(t)\overline{h}_2(t)]^{n+1} t^{2n+1}}{n!} \int_{0}^{t} z_0(\tau)d\tau 
\] (5.14)

where

\[
\overline{h}_1(t) = \text{ess sup}\{h_1(s) : 0 \leq s \leq t\}, \\
\overline{h}_2(t) = \text{ess sup}\{h_2(s, \tau) : 0 \leq \tau \leq s \leq t\}.
\]

Obviously \(\overline{h}_1(t), \overline{h}_2(t)\) are nondecreasing functions. Now, we apply the measure of noncompactness \(\gamma_R\) defined on \(C(\mathbb{R}_+, E)\) by [5.3]. Let

\[
R(t) = r(t) \left( 1 + \int_{0}^{t} z_0(s)ds \right) \exp(M\overline{h}_1(t)\overline{h}_2(t)t^2)
\]

Observe that \(R(t) \geq r(t)\). Then, in view of definition of \(\mu^T\) given by [5.2], we have

\[
\mu^T(Q_{n+1}) = \sup_{t \leq T} \{ z_{n+1}(t) \} \leq \frac{(M\overline{h}_1(T)\overline{h}_2(T))^{n+1} T^{2n+1}}{n!} \int_{0}^{T} z_0(s)ds.
\]

By using the estimation

\[
\sup\left\{ \frac{t^n}{n! \exp(t)} : t \geq 0 \right\} \leq \frac{n^n}{n! \exp(n)}, \quad \text{for all } n \in \mathbb{N},
\]

we get

\[
\frac{\mu^T(Q_{n+1})}{R(T)} \leq \frac{(M\overline{h}_1(T)\overline{h}_2(T)T^2)^n T}{r(T)n! \exp(M\overline{h}_1(T)\overline{h}_2(T)T^2)} \leq \frac{n^{n+1}}{r(T)n! \exp(n)}.
\]

Combining [5.11] and the above inequality, we have

\[
\lim_{n \to \infty} \gamma_R(Q_{n+1}) = \gamma_R(Q_{\infty}) = 0.
\]
Under the properties of \( \gamma_R \) given by theorem 5.1, \( Q_\infty \) is nonempty subset, closed and convex. By applying the Tichonov fixed point theorem for \( F : Q_\infty \to Q_\infty \) we conclude that \( F \) has at least fixed point \( x \in Q_\infty \). Obviously, the function \( x \) is a mild solution of (5.1).

**Case 2:** \( \mu(Q_{n-1}(s)) \geq \mu\left( \int_0^s u(s, \tau, z_n(Q_{n-1}(\tau)))d\tau \right) \). The proof of the existence of mild solution of (5.1) can be done by using the same techniques as in the case 1. This completes the proof.
Bibliography


